

Nonperturbative Functional Renormalization Group for Random Field Models. IV: Supersymmetry and its spontaneous breaking.

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We apply the nonperturbative functional renormalization group (NP-FRG) in the superfield formalism that we have developed in the preceding paper to study long-standing issues concerning the critical behavior of the random field Ising model. Through the introduction of an appropriate regulator and a supersymmetry-compatible nonperturbative approximation, we are able to follow the supersymmetry, more specifically the superrotational invariance first unveiled by Parisi and Sourlas [Phys. Rev. Lett. **43**, 744 (1979)], and its spontaneous breaking along the RG flow. Breaking occurs below a critical dimension $d_{DR} \simeq 5.1$, and the supersymmetry-broken fixed point that controls the critical behavior then leads to a breakdown of the “dimensional reduction” property. We solve the NP-FRG flow equations numerically and determine the critical exponents as a function of dimension down to $d \lesssim 3$, with a good agreement in $d = 3$ and $d = 4$ with the existing numerical estimates.

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I. INTRODUCTION

Quenched disorder induces an extrinsic inhomogeneity in otherwise translationally invariant pure systems. As a result, the equilibrium and out-of-equilibrium physics of disordered systems may be influenced by rare collective events, such as “avalanches”, statistically unlikely regions, such as “droplets” or “Griffiths regions”, and by the proliferation of “metastable states”. Our objective is to develop a theory describing the long-distance physics of such systems through a nonperturbative functional renormalization group (NP-FRG) method and our first focus is the equilibrium behavior of the random-field model. We showed in previous papers, which we refer to as I¹ and II², that the effect of avalanches and droplets can be captured by an approach based on the RG flow of the cumulants of the renormalized disorder, provided that the full functional dependence of the latter is accounted for.³ A functional RG is therefore required to let the singular behavior due to rare events or regions emerge along the flow, as also shown in the case of the equilibrium and forced behavior of manifolds in a random environment.^{4–11}

The issue of metastable states, which, at zero temperature where the concept is well defined, refers to the presence of many minima of the microscopic hamiltonian (bare action) not simply related by symmetry transformations, is a recurring conundrum in theories of disordered systems. In the case of the random-field Ising model (RFIM) under study, it has a clear manifestation. The critical behavior of the model being controlled by a zero-temperature fixed point,^{12–14} the long-distance physics can be described through the properties of the ground state which, due to the presence of the random field, is obtained as the solution of a stochastic field equation. By standard field-theoretic manipulations, this leads to a theory expressed in terms of superfields. Parisi and Sourlas¹⁵ showed that a supersymmetry of the the-

ory, more specifically the invariance under rotations of the underlying superspace, implies the property of dimensional reduction, according to which the critical behavior of the RFIM in d dimensions is identical to that of the pure Ising model in $d - 2$ dimensions. The property however has been proven to be wrong in low enough dimension.^{16,17} At the same time, it has been understood that the superfield construction breaks down because of the presence of many solutions of the stochastic field equation.¹⁸ However, in the Parisi-Sourlas formalism, it is not possible to disentangle breaking of superrotational invariance and collapse of the formalism due to the appearance of multiple solutions.

In the companion paper, henceforth referred to as paper III,¹⁹ we have shown how to resolve the above conundrum and to combine an extended superfield approach with the NP-FRG formalism. In particular, ground-state selection can be achieved by adding a weighting factor involving an auxiliary temperature and letting at the end of the manipulations the auxiliary temperature go to zero in the exact NP-FRG equations for the cumulants of the renormalized disorder. The resulting property of “Grassmannian ultralocality” then allows one to specifically investigate supersymmetry (superrotational invariance) and its spontaneous breaking along the RG flow. Such an investigation is the purpose of the present paper.

The outline of the article is as follows. In Sec. II, we briefly summarize the main steps of the NP-FRG in a superfield formalism that we have developed in paper III. This allows us to recall definitions, notations, and RG equations that will be used in the present article. In particular, we stress two important, and distinct, formal properties of the superfield theory: “Grassmannian ultralocality” and “superrotational invariance”. We also consider the RG flow of the Ward-Takahashi identities associated with the latter and the consequences for the choice of the infrared regulator. In Sec. III, we show through our NP-FRG formalism that the superrotational

invariance nonperturbatively leads to dimensional reduction. We conclude the section by building a scenario for a spontaneous breaking of the superrotational invariance along the RG flow that is based on our previous results on the appearance of a “cusp” in the functional dependence of the cumulants of the renormalized random field,^{1,2,20,21} and we propose a continuation of the NP-FRG flow equations when spontaneous breaking has taken place.

Sec. IV is devoted to the development of a supersymmetry-compatible nonperturbative approximation scheme for the exact NP-FRG equations. It relies on combined truncations of the cumulant expansion and the expansion in spatial derivatives of the field. We also provide details on the numerical resolution. In Sec. V we present the results. We show in particular that breakdown of the dimensional reduction predictions for the critical exponents of the RFIM occurs below a critical dimension $d_{DR} \simeq 5.1$. We compute the critical exponents as a function of dimension down to $d = 3$ and we find good agreement with the best available estimates in $d = 3$ and $d = 4$. Finally, we show that scaling is described by three independent exponents, contrary to a proposed conjecture.^{22–24}

A short account of this work has appeared in Ref.[25].

II. NP-FRG IN THE SUPERFIELD FORMALISM FOR THE RFIM

A. Summary

We start by briefly recalling the main features of the formalism presented in the preceding article, with the associated definitions and notations.

In paper III,¹⁹ we have developed a NP-FRG theory for describing the equilibrium long-distance physics of the RFIM that is based on an extension of the Parisi-Sourlas¹⁵ supersymmetric formalism. The latter relies on the fact that the critical behavior of the model is dominated by disorder-induced fluctuations, thermal fluctuations being subdominant, and can therefore be studied by looking directly at zero temperature. The equilibrium properties are then described by the ground state which is solution of the following stochastic field equation

$$\frac{\delta S[\varphi; h]}{\delta \varphi(x)} = J(x), \quad (1)$$

where we have added an external source (a magnetic field) J conjugate to the φ field and the action $S[\varphi; h]$ is given by

$$S = \int_x \left\{ \frac{1}{2} (\partial_\mu \varphi(x))^2 + U_B(\varphi(x)) - h(x)\varphi(x) \right\}, \quad (2)$$

where $\int_x \equiv \int d^d x$, $U_B(\varphi) = (\tau/2)\varphi^2 + (u/4!)\varphi^4$, with $h(x)$ a random (magnetic) field sampled from a Gaussian distribution of zero mean and variance $\overline{h(x)h(y)} = \Delta_B \delta^{(d)}(x - y)$.

Two key ingredients of our extension of the Parisi-Sourlas formalism are:

(1) the need to consider multiple copies (or replicas) of the original system with the same disorder, each copy being coupled to a different applied source, in order to generate cumulants of the renormalized disorder with their full functional dependence, thereby allowing for the emergence of a nonanalytic behavior in the field arguments,

(2) the introduction of a weighting factor involving an auxiliary temperature β^{-1} to the solutions of the stochastic field equation, so that when β^{-1} approaches 0 only the ground state contributes to the generating functional.

By using standard field-theoretical techniques,²⁶ one then ends up with a superfield theory for multiple copies in a curved superspace, with

$$\begin{aligned} \mathcal{Z}^{(\beta)}[\{\mathcal{J}_a\}] &= \exp(\mathcal{W}^{(\beta)}[\{\mathcal{J}_a\}]) = \overline{\exp\left(\sum_{a=1}^n \mathcal{W}_h^{(\beta)}[\mathcal{J}_a]\right)} \\ &= \int \prod_{a=1}^n \mathcal{Q}\Phi_a \exp\left(-S^{(\beta)}[\{\Phi_a\}] + \sum_{a=1}^n \int_{\underline{x}} \mathcal{J}_a(\underline{x})\Phi_a(\underline{x})\right), \end{aligned} \quad (3)$$

where the multicopy action is given by

$$\begin{aligned} S^{(\beta)}[\{\Phi_a\}] &= \sum_{a=1}^n \int_{\underline{x}} \left[\frac{1}{2} (\partial_\mu \Phi_a(\underline{x}))^2 + U_B(\Phi_a(\underline{x})) \right] \\ &\quad - \frac{\Delta_B}{2} \sum_{a_1=1}^n \sum_{a_2=1}^n \int_x \int_{\underline{\theta}_1 \underline{\theta}_2} \Phi_{a_1}(x, \underline{\theta}_1) \Phi_{a_2}(x, \underline{\theta}_2). \end{aligned} \quad (4)$$

In the above equations, we have introduced a superspace (coordinates \underline{x}) comprising the d -dimensional Euclidean space (coordinates x) and a 2-dimensional Grassmannian space (anticommuting coordinates $\underline{\theta} = \{\theta, \bar{\theta}\}$); the metric is flat in the Euclidean sector and curved in the Grassmannian one with the curvature proportional to β . (For instance, the integral over superspace is defined as $\int_{\underline{x}} \equiv \int_x \int_{\underline{\theta}} \equiv \int d^d x \int \int (1 + \beta \bar{\theta} \theta) d\theta d\bar{\theta}$.) We have also defined superfields $\Phi_a(\underline{x}) = \varphi_a(x) + \bar{\theta} \psi_a(x) + \bar{\psi}_a(x) \theta + \bar{\theta} \theta \hat{\phi}_a(x)$, with one auxiliary bosonic (“response”) field $\hat{\phi}_a$ and two auxiliary fermionic (“ghost”) fields $\bar{\psi}_a$ and ψ_a , and associated supersources $\mathcal{J}_a(\underline{x}) = J_a(x) + \bar{\theta} K_a(x) + \bar{K}_a(x) \theta + \bar{\theta} \theta \hat{J}_a(x)$. The action in Eq. (4) is invariant under a large group of (bosonic and fermionic) symmetries.

When expanded in increasing number of sums over copies, the generating functional of the connected Green’s functions, $\mathcal{W}^{(\beta)}[\{\mathcal{J}_a\}]$, gives access to the cumulants of the random generating functional $\mathcal{W}_h^{(\beta)}[\mathcal{J}]$.

We have next applied the NP-FRG formalism to this superfield theory. This proceeds by first adding an infrared (IR) regulator that enforces a progressive account

of the fluctuations to the bare action,

$$\begin{aligned} \Delta S_k^{(\beta)} = & \frac{1}{2} \sum_a \int_{x_1 x_2} \int_{\underline{\theta}} \Phi_a(x_1, \underline{\theta}) \hat{R}_k(|x_1 - x_2|) \Phi_a(x_2, \underline{\theta}) \\ & + \frac{1}{2} \sum_{a_1, a_2} \int_{\underline{x}_1 \underline{x}_2} \Phi_{a_1}(\underline{x}_1) \tilde{R}_k(|x_1 - x_2|) \Phi_{a_2}(\underline{x}_2), \end{aligned} \quad (5)$$

where the two cutoff functions \hat{R}_k and \tilde{R}_k are related through

$$\tilde{R}_k(q^2) = -\frac{\Delta_k}{Z_k} \partial_{q^2} \hat{R}_k(q^2), \quad (6)$$

with q the Euclidean momentum, Δ_k the strength of the renormalized random field, and Z_k the field renormalization constant. This ensures that all symmetries of the theory are satisfied. This includes the superrotational invariance found when the theory is restricted to a single copy and to $\beta = 0$: it then corresponds to $\Delta_k = \Delta_B Z_k$, a property that is valid at the microscopic (UV) scale Λ (see also below).

One next introduces the effective average action^{27,28} $\Gamma_k^{(\beta)}[\{\Phi_a\}] = -\mathcal{W}_k^{(\beta)}[\{\mathcal{J}_a\}] + \sum_a \int_{\underline{x}} \Phi_a(\underline{x}) \mathcal{J}_a(\underline{x}) - \Delta S_k^{(\beta)}[\{\Phi_a\}]$, whose dependence on the IR cutoff k is governed by an exact renormalization-group equation (ERGE). The effective average action can also be expanded in increasing number of sums over copies, each term of the expansion being then related through the Legendre transform to the cumulants of $\mathcal{W}_h^{(\beta)}[\mathcal{J}]$. We refer to the p th order term of the expansion of $\Gamma_k^{(\beta)}[\{\Phi_a\}]$ as the p th “cumulant of the renormalized disorder”. These expansions in increasing number of sums over copies lead to systematic algebraic manipulations that allow one to derive from the ERGE for $\Gamma_k^{(\beta)}[\{\Phi_a\}]$ a hierarchy of coupled ERGE’s for the cumulants of the renormalized disorder.

We have unveiled an important property of the random generating functional $\mathcal{W}_h^{(\beta)}[\mathcal{J}]$: the latter satisfies a specific dependence on the Grassmann coordinates, which we have called “Grassmannian ultralocality”, if and only if a unique solution of the stochastic field equation is included in its computation. This translates into an “ultralocal” property of the cumulants of the renormalized disorder. “Grassmannian ultralocality” becomes a property of the superfield theory when $\beta \rightarrow \infty$; it is also asymptotically found for finite β when $k \rightarrow 0$ (after going to dimensionless quantities). It then reflects the desired ground-state dominance. When this property is satisfied (e.g. by setting $\beta^{-1} = 0$), the ERGE’s for the cumulants simplify and only involve “ultralocal” quantities that can be evaluated for physical fields $\Phi_a(\underline{x}) = \phi_a(x)$, i.e. for superfields that are uniform in the Grassmann subspace.

For illustration, we give below the ERGE’s for the first two cumulants under the property of “Grassmannian ul-

tralocality”, which will be needed in the following:

$$\begin{aligned} \partial_t \Gamma_{k1}[\phi_1] = & -\frac{1}{2} \tilde{\partial}_t \int_{x_2 x_3} \hat{P}_{k;x_2 x_3}[\phi_1] (\Gamma_{k2;x_2, x_3}^{(11)}[\phi_1, \phi_1] - \tilde{R}_{k;x_2 x_3}) \end{aligned} \quad (7)$$

and

$$\begin{aligned} \partial_t \Gamma_{k2}[\phi_1, \phi_2] = & \frac{1}{2} \tilde{\partial}_t \int_{x_3 x_4} \{ -\hat{P}_{k;x_3 x_4}[\phi_1] \Gamma_{k2;x_3, \dots, x_4}^{(101)}[\phi_1, \phi_2, \phi_1] + \\ & \tilde{P}_{k;x_3 x_4}[\phi_1, \phi_1] \Gamma_{k2;x_3 x_4}^{(20)}[\phi_1, \phi_2] + \frac{1}{2} \tilde{P}_{k;x_3 x_4}[\phi_1, \phi_2] \\ & \times (\Gamma_{k2;x_3, x_4}^{(11)}[\phi_1, \phi_2] - \tilde{R}_{k;x_3 x_4}) + perm(12) \}, \end{aligned} \quad (8)$$

where $perm(12)$ denotes the expression obtained by permuting ϕ_1 and ϕ_2 , $t = \log(k/\Lambda)$, $\tilde{\partial}_t$ is a short-hand notation to indicate a derivative acting only on the cutoff functions (i.e., $\tilde{\partial}_t \equiv \partial_t \hat{R}_k \delta/\delta \hat{R}_k + \partial_t \tilde{R}_k \delta/\delta \tilde{R}_k$), and the propagators \hat{P}_k and \tilde{P}_k (denoted by $\hat{P}_k^{[0]}$ and $\tilde{P}_k^{[0]}$ in paper III¹⁹) are defined as

$$\hat{P}_k[\phi] = \left(\Gamma_{k,1}^{(2)}[\phi] + \hat{R}_k \right)^{-1} \quad (9)$$

and

$$\tilde{P}_k[\phi_1, \phi_2] = \hat{P}_k[\phi_1] (\Gamma_{k,2}^{(11)}[\phi_1, \phi_2] - \tilde{R}_k) \hat{P}_k[\phi_2], \quad (10)$$

and superscripts indicate functional differentiation with respect to the field arguments.

Generically, the flow of $\Gamma_{kp}[\phi_1, \dots, \phi_p]$ involves three types of quantities: the propagators \hat{P}_k and \tilde{P}_k , second functional derivatives of Γ_{kp} in which all the arguments are different, and second functional derivatives of $\Gamma_{k(p+1)}$ with two of their arguments equal to each other. A graphical representation of the hierarchy of ERGE’s is provided in Appendix C of the companion paper III.

In the present paper we will focus on these flow equations, which correspond to the $\beta \rightarrow \infty$ limit and describe the equilibrium properties associated with the ground state (recall that the “bath” temperature is also at $T = 0$). We will briefly comment on the RG flow of corrections to “Grassmannian ultralocality” when β is large but finite.

Finally, it is worth stressing that to obtain the flow equation for $\Gamma_{kp}[\phi_1, \dots, \phi_p]$ with its full functional dependence on the p field arguments, one needs to consider at least p copies. If one considers less than p copies, the equation necessarily involves Γ_{kp} and its derivatives in which several of the arguments are equal. As we will show later, breaking of supersymmetry and breakdown of dimensional reduction are precisely related to the analyticity properties of the functionals when taking the limit of equal arguments. Formally, the whole hierarchy of flow equations for the cumulants can thus be obtained by considering an arbitrary large number of copies.

B. Superrotational invariance, Ward-Takahashi identities and RG flow

When “Grassmannian ultralocality” is satisfied, the curvature β disappears from the flow equations [see Eqs. (7,8)]. Formally, the latter are then the same as those obtained from considering a flat Grassmann subspace. We have shown in paper III that in this case, and provided one restricts the supersources such that one effectively recovers a one-copy theory, the theory is invariant under “superrotations” that mix the Euclidean and Grassmannian sectors of the superspace. The associated generators are $\overline{\mathcal{Q}}_\mu = -x_\mu \partial_\theta + (2/\Delta_B) \bar{\theta} \partial_\mu$ and $\mathcal{Q}_\mu = x_\mu \partial_{\bar{\theta}} + (2/\Delta_B) \theta \partial_\mu$. As any linearly realized continuous symmetry, superrotational invariance leads to a set of Ward-Takahashi (WT) identities for the one-particle irreducible (1PI) generating functional, *i.e.* at scale k the effective average action. For a flat superspace and a restriction to one copy, the WT identity for the effective average action reads

$$\int_{\underline{x}} \Phi(\underline{x}) \mathcal{Q}_\mu \Gamma_{k;\underline{x}}^{(1)}[\Phi] = 0 \quad (11)$$

and similarly with $\overline{\mathcal{Q}}_\mu$.

One can now check that the above WT identities are *a priori* stable under the RG flow, *i.e.*

$$\partial_t \int_{\underline{x}} \Phi(\underline{x}) \mathcal{Q}_\mu \Gamma_{k;\underline{x}}^{(1)}[\Phi] = 0, \quad (12)$$

where $\mathcal{Q}_{\underline{x}}$ generically indicates a component \mathcal{Q}_μ and, since there is no curvature, $\int_{\underline{x}} \equiv \int d^d x \int d\theta d\bar{\theta}$. To prove Eq. (12), we rewrite the ERGE for the effective average action when the superfield theory is restricted to one copy and is considered in a flat superspace ($\beta = 0$):

$$\partial_t \Gamma_k[\{\Phi_a\}] = \frac{1}{2} \int_{\underline{x}_1 \underline{x}_2} (\partial_t \mathcal{R}_{k;\underline{x}_1, \underline{x}_2}) \mathcal{P}_{k;\underline{x}_1, \underline{x}_2}[\{\Phi_a\}], \quad (13)$$

where the regulator function $\mathcal{R}_{k;\underline{x}_1, \underline{x}_2} = \delta_{\bar{\theta}_1, \bar{\theta}_2} \hat{R}_k(|x_1 - x_2|) + \tilde{R}_k(|x_1 - x_2|)$ with $\delta_{\bar{\theta}_1, \bar{\theta}_2} = (\bar{\theta}_1 - \bar{\theta}_2)(\theta_1 - \theta_2)$ and the modified (full) propagator \mathcal{P}_k is the inverse in the sense of operators of $\Gamma_k^{(2)} + \mathcal{R}_k$. (Inserting the “multilocal” expansion in the Grassmann coordinates and using the assumption of “Grassmannian ultralocality” then leads to ERGE’s for the cumulants which are just Eqs. (7,8) and their extension to higher orders with all copy superfields taken as equal in both sides of the equations.)

The flow of the left-hand side of Eq. (12) can then be expressed as

$$\begin{aligned} & \int_{\underline{x}} \Phi(\underline{x}) \mathcal{Q}_\mu \partial_t \Gamma_{k;\underline{x}}^{(1)}[\Phi] \\ &= \frac{1}{2} \tilde{\partial}_t \int_{\underline{x}_1 \underline{x}_2 \underline{x}_3} \Phi(\underline{x}_1) \mathcal{Q}_{\underline{x}_1} \left(\mathcal{P}_{k;\underline{x}_2 \underline{x}_3}[\Phi] \Gamma_{k;\underline{x}_3 \underline{x}_2 \underline{x}_1}^{(3)}[\Phi] \right) \\ &= \frac{1}{2} \tilde{\partial}_t \int_{\underline{x}_2 \underline{x}_3} \mathcal{P}_{k;\underline{x}_2 \underline{x}_3}[\Phi] \int_{\underline{x}_1} \Phi(\underline{x}_1) \mathcal{Q}_{\underline{x}_1} \Gamma_{k;\underline{x}_3 \underline{x}_2 \underline{x}_1}^{(3)}[\Phi], \end{aligned} \quad (14)$$

where we have used that $\mathcal{Q}_{\underline{x}_1}$ does not act on $\mathcal{P}_{k;\underline{x}_2 \underline{x}_3}$. Assuming that the WT identity, Eq. (11), is satisfied down to the scale k (we study its further evolution), one has

$$\int_{\underline{x}_1} \Phi(\underline{x}_1) \mathcal{Q}_{\underline{x}_1} \Gamma_{k;\underline{x}_3 \underline{x}_2 \underline{x}_1}^{(3)}[\Phi] = -(\mathcal{Q}_{\underline{x}_2} + \mathcal{Q}_{\underline{x}_3}) \Gamma_{k;\underline{x}_3 \underline{x}_2}^{(2)}[\Phi], \quad (15)$$

which is obtained by differentiating twice Eq. (11). After an integration by parts and a relabel of the dummy variables $\underline{x}_2, \underline{x}_3$, one then finds

$$\begin{aligned} & \int_{\underline{x}} \Phi(\underline{x}) \mathcal{Q}_\mu \partial_t \Gamma_{k;\underline{x}}^{(1)}[\Phi] \\ &= -\frac{1}{2} \tilde{\partial}_t \int_{\underline{x}_2 \underline{x}_3} \Gamma_{k;\underline{x}_3 \underline{x}_2}^{(2)}[\Phi] (\mathcal{Q}_{\underline{x}_2} + \mathcal{Q}_{\underline{x}_3}) \mathcal{P}_{k;\underline{x}_2 \underline{x}_3}[\Phi]. \end{aligned} \quad (16)$$

Provided that one chooses the infrared cutoff function such that

$$(\mathcal{Q}_{\underline{x}_1} + \mathcal{Q}_{\underline{x}_2}) \mathcal{R}_{k;\underline{x}_1 \underline{x}_2} = 0, \quad (17)$$

one can replace $\Gamma_{k;\underline{x}_3 \underline{x}_2}^{(2)}[\Phi]$ by $\Gamma_{k;\underline{x}_3 \underline{x}_2}^{(2)}[\Phi] + \mathcal{R}_{k;\underline{x}_3 \underline{x}_2}$ in Eq. (16). By using the fact that the latter is equal to the inverse modified propagator $(\mathcal{P}_k^{-1}[\Phi])_{\underline{x}_3 \underline{x}_2}$, one can rewrite the right-hand side of Eq. (16) (up to the trivial factor of $-1/2$) as

$$\begin{aligned} & \tilde{\partial}_t \int_{\underline{x}_2 \underline{x}_3} (\mathcal{P}_k^{-1}[\Phi])_{\underline{x}_3 \underline{x}_2} \mathcal{Q}_{\underline{x}_2} \mathcal{P}_{k;\underline{x}_2 \underline{x}_3}[\Phi] \\ &= \tilde{\partial}_t \int_{\underline{x}_2 \underline{x}_4} \delta_{\underline{x}_2 \underline{x}_4} \int_{\underline{x}_3} (\mathcal{P}_k^{-1}[\Phi])_{\underline{x}_3 \underline{x}_4} \mathcal{Q}_{\underline{x}_2} \mathcal{P}_{k;\underline{x}_2 \underline{x}_3}[\Phi] \\ &= \tilde{\partial}_t \int_{\underline{x}_2 \underline{x}_4} \delta_{\underline{x}_2 \underline{x}_4} \mathcal{Q}_{\underline{x}_2} \int_{\underline{x}_3} (\mathcal{P}_k^{-1}[\Phi])_{\underline{x}_3 \underline{x}_4} \mathcal{P}_{k;\underline{x}_2 \underline{x}_3}[\Phi] \\ &= \tilde{\partial}_t \int_{\underline{x}_2 \underline{x}_4} \delta_{\underline{x}_2 \underline{x}_4} \mathcal{Q}_{\underline{x}_2} \delta_{\underline{x}_2 \underline{x}_4}. \end{aligned} \quad (18)$$

This last expression is easily shown to be identically zero, so that one finally obtains that Eq. (12) is satisfied at scale k . The same reasoning can be repeated with the identity associated with $\overline{\mathcal{Q}}_\mu$.

Therefore, if one starts with an initial condition that is superrotationally invariant, which can be enforced by suppressing fluctuations (see section II-A and paper III), the above derivation proves that the WT identities associated with the superrotational invariance are *a priori* preserved along the RG flow if the regulator satisfies Eq. (17). The latter condition can be reexpressed as

$$\begin{aligned} & (\mathcal{Q}_{1\mu} + \mathcal{Q}_{2\mu}) \mathcal{R}_{k;\underline{x}_1 \underline{x}_2} = (\theta_1 - \theta_2) \left[(x_{1\mu} - x_{2\mu}) \hat{R}_k(|x_1 - x_2|) \right. \\ & \left. + \frac{2}{\Delta_B} \partial_{1\mu} \tilde{R}_k(|x_1 - x_2|) \right] = 0, \end{aligned} \quad (19)$$

where we have used the fact that \hat{R}_k and \tilde{R}_k are translationally invariant functions of the Euclidean coordinates,

with therefore $\partial_{1\mu}\hat{R}_k = -\partial_{2\mu}\hat{R}_k$ and similarly for \tilde{R}_k . Going to Fourier space, this implies that \hat{R}_k and \tilde{R}_k are related through

$$\tilde{R}_k(q^2) = -\Delta_B \partial_{q^2} \hat{R}_k(q^2), \quad (20)$$

which, as already stated, corresponds to Eq. (6) with $\Delta_k = \Delta_B Z_k$, a condition that we choose to enforce at the beginning of the flow when $k = \Lambda$. Such a choice of cutoff then avoids an *explicit* breaking of superrotational invariance.

III. SUPERROTATIONAL INVARIANCE AND ITS SPONTANEOUS BREAKING

A. SUSY and dimensional reduction

An important property of the theory is that the superrotational invariance (which for simplicity will often be denoted by the acronym SUSY, for supersymmetry, in the following) leads to dimensional reduction. We consider here the case where “Grassmannian ultralocality” is satisfied so that the curvature β drops out of the flow equations as in Eqs. (7,8), and we assume that SUSY is indeed obeyed when the theory is restricted to one copy (see above). We showed in paper III¹⁹ that SUSY implies nontrivial WT identities relating cumulants of different orders. More specifically, the following WT identity at the scale k will be needed below:

$$\begin{aligned} \partial_{1\mu} \Gamma_{k2;x_1,x_2}^{(11)}[\phi, \phi] - \frac{\Delta_B}{2} (x_1^\mu - x_2^\mu) \Gamma_{k1;x_1,x_2}^{(2)}[\phi] = \\ - \int_{x_3} \phi(x_3) \partial_{3\mu} \Gamma_{k2;x_1,x_3,x_2}^{(21)}[\phi, \phi]. \end{aligned} \quad (21)$$

To prove the dimensional reduction property, it is then useful to single out a 2-dimensional subspace of the d -dimensional Euclidean space. We define $x = (y, z)$ with $y \in R^{d-2}$ and $z \in R^2$ and we consider superfields that are uniform in the 2-dimensional Grassmannian and the 2-dimensional Euclidean subspaces, *i.e.* $\Phi(\underline{x}) = \phi(y)$. For such fields, the WT identity in Eq. (21) implies that

$$\Gamma_{k2;y_1,y_2}^{(11)}[p^2; \phi, \phi] = \Delta_B \partial_{p^2} \Gamma_{k1;y_1,y_2}^{(2)}[p^2; \phi], \quad (22)$$

where p is the 2-dimensional momentum obtained from a spatial Fourier transform over z [by using translational invariance we have as usual defined $\Gamma_{k1}^{(2)}(p_1, p_2) = (2\pi)^2 \delta^{(2)}(p_1 + p_2) \Gamma_{k1}^{(2)}(p^2)$ and similarly for $\Gamma_{k2}^{(11)}(p_1, p_2)$]. The same type of relation holds between $-\hat{R}_{k;y_1,y_2}(p^2)$ and $\hat{R}_{k;y_1,y_2}(p^2)$.

With the help of the above relations, the ERGE for

the first cumulant $\Gamma_{k1}[\phi]$, Eq. (7), can be reexpressed as

$$\begin{aligned} \partial_t \Gamma_{k1}[\phi] = & -\left(\frac{\Delta_B}{2}\right) \tilde{\partial}_t \int d^{d-2} y_1 \int d^{d-2} y_2 \int \frac{d^2 p}{(2\pi)^2} \\ & \times \hat{P}_{k;y_1,y_2}[p^2, \phi] \partial_{p^2} \left(\Gamma_{k1;y_1,y_2}^{(2)}[p^2, \phi] + \hat{R}_k(p^2) \right) \\ = & -\left(\frac{\Delta_B}{2}\right) \tilde{\partial}_t \int d^{d-2} y_1 \int \frac{d(p^2)}{4\pi} \\ & \times \partial_{p^2} \left[\log \left(\Gamma_{k1}^{(2)}[p^2, \phi] + \hat{R}_k(p^2) \right) \right]_{y_1 y_1}, \end{aligned} \quad (23)$$

where, in the last expression, $\Gamma_{k1}^{(2)}$ and \hat{R}_k are functions of p^2 but operators in the $(d-2)$ -dimensional Euclidean space spanned by the coordinate y (since $\phi(y)$ is *not* uniform). The integral over p^2 is easily performed. Choosing an infrared cutoff function that becomes independent of k when its argument p^2 is at the UV scale, it only remains:

$$\begin{aligned} \partial_t \Gamma_{k1}[\phi] = & \left(\frac{\Delta_B}{4\pi}\right) \frac{1}{2} \tilde{\partial}_t \int d^{d-2} y \\ & \times \left[\log \left(\Gamma_{k1}^{(2)}[p^2 = 0, \phi] + \hat{R}_k(p^2 = 0) \right) \right]_{yy}. \end{aligned} \quad (24)$$

Up to the trivial factor $\Delta_B/(4\pi)$ and with the identifications

$$\Gamma_{k1}[\phi] \equiv \Gamma_k[\phi], \Gamma_{k1}^{(2)}[p^2 = 0, \phi] \equiv \Gamma_k^{(2)}[\phi], \hat{R}_k(p^2 = 0) \equiv R_k, \quad (25)$$

Eq. (24) coincides with the ERGE for the effective average action of a standard scalar ϕ^4 theory²⁸ in dimension $d-2$. This provides another nonperturbative demonstration²⁹⁻³¹ of the property of dimensional reduction for supersymmetric scalar field theories, first put forward by Parisi and Sourlas.¹⁵

B. Spontaneous SUSY breaking

We have shown within a nonperturbative implementation of the RG that the supersymmetry of the theory, and more specifically the superrotational invariance for one copy (SUSY), imply the property of dimensional reduction. As one knows that dimensional reduction does not hold in low enough dimension, what then goes wrong in the formalism?

The answer is that some (super)symmetries or identities must be spontaneously broken along the RG flow. We make the assumption, which will be supported by actual calculations, that only the superrotational invariance (for the theory restricted to a single copy) may be broken along the RG flow, all other (super)symmetries and properties, most significantly the “Grassmannian ultralocality” encoding ground-state dominance (when $\beta \rightarrow \infty$), remaining unaltered. From the above proof, it follows that failure of dimensional reduction implies that the WT identity between $\Gamma_{k2}^{(11)}$ and $\Gamma_{k1}^{(2)}$ breaks down: a singularity must occur along the flow, which invalidates the WT identity; the equality in Eq. (12) and some step in its

derivation must go wrong at some scale k . From our previous work,^{1,2,20,21} we anticipate that this arises due to the appearance of a strong enough nonanalytic dependence of the second cumulant of the renormalized random field $\Gamma_{k2}^{(11)}[\phi_1, \phi_2]$ in its field arguments when the latter two, ϕ_1 and ϕ_2 , become equal. (One should keep in mind that the cumulants are invariant under permutations of their arguments; consequently, $\Gamma_{k2}^{(11)}$ is an even function(al) in $\phi_1 - \phi_2$.) As a result, some higher-order derivative (*i.e.*, a 1PI vertex deriving from $\Gamma_{k2}^{(11)}$) blows up and the whole hierarchy of coupled ERGE's and WT identities for one copy ceases to be valid.

In order to envisage the possible scenario for spontaneous SUSY breaking and breakdown of dimensional reduction, it is instructive to study the structure of the exact hierarchy of coupled RG flow equations, whose lowest-order examples are given in Eqs. (7,8). The following exposition is not meant to be rigorous, but only to provide some heuristic arguments.

Consider first the hierarchy of ERGE's for the proper vertices evaluated for uniform fields (in Euclidean space) that is obtained by repeated functional differentiation of the ERGE for the effective average action. For this part of the reasoning, the case of a simple scalar field theory is sufficiently illustrative. The ERGE for the effective average action then reads

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int_q \partial_t R_k(q^2) \left(\Gamma_k^{(2)}[\phi] + R_k \right)_{q, -q}^{-1}. \quad (26)$$

After differentiating this equation s times, it is easily realized that for $s \geq 2$ the ERGE for any generic 1PI vertex $\Gamma_k^{(s)}$ depends on all lower-order proper vertices (of order ≥ 2) as well as on some $\Gamma_k^{(s+1)}$'s and $\Gamma_k^{(s+2)}$'s. Two cases then arise: when $s \geq 5$, the flow equation is *linear* in $\Gamma_k^{(s)}$ itself, whereas in the other case when $4 \geq s \geq 2$, the flow equation may be *nonlinear* in $\Gamma_k^{(s)}$ itself. The same structure applies to the present theory. A consequence is that, if no other 1PI vertices diverge first, $\Gamma_k^{(s)}$ can only diverge at the end of the flow ($k \rightarrow 0$) when $s \geq 5$ (due to the linearity of the corresponding ERGE) whereas a divergence may occur at a finite scale when $s \leq 4$.

To get more insight, we have to look more thoroughly into the structure of the RG equations and consider the 1PI vertices associated with the cumulants separately [see Eqs. (7,8) for illustration]. We further make the following plausible assumptions (in order to have a renormalizable theory):

(i) The 1-copy 1PI vertices $\Gamma_{k1}^{(s)}$ always stay finite, as do the cumulants of the renormalized random field $\Gamma_{kp}^{(11\dots 1)}$ and their first-order derivatives. The latter may however be discontinuous; the condition on the first derivatives amounts to assuming that no “supercusp”³² stronger than linear appears in the cumulants when two arguments become equal.

(ii) Since we expect a nonanalytic behavior occurring as the copy fields become equal in the reduction

to an effective 1-copy system, we also anticipate that derivatives with respect to symmetric combinations of the field arguments are always bounded (*e.g.*, $|\partial\phi_1 + \partial\phi_2| \Gamma_{kp}^{(s_1 s_2 \dots s_p)}[\phi_1, \phi_2, \dots, \phi_p] | \phi| < +\infty$, where the derivative is evaluated for all replica fields equal to ϕ).

A “cusp”³² in $\Gamma_{kp}^{(11\dots 1)}$ with $p \geq 3$ implies that one of its second derivatives, which means a proper vertex of order at least $s = 5$, blows up. Similarly, for having a “subcusp”³² in $\Gamma_{k2}^{(11)}$, at least one of its third derivatives, hence a proper vertex of order at least $s = 5$, should blow up. As we have seen above that the associated flow equations are linear, we conclude that the divergence of such proper vertices (with $s \geq 5$) cannot be the triggering event for a singularity at a finite scale. On the other hand, due to the nonlinearity of the ERGE's for the second derivatives of $\Gamma_{k2}^{(11)}$, a “cusp” may appear in the latter at a finite scale.

To summarize this discussion: one expects that breakdown of dimensional reduction requires the presence of a *cusp* in the field dependence of $\Gamma_{k2}^{(11)}$, cusp that should appear *at a finite scale*, which by analogy with what occurs for a manifold in a random environment^{9,10,33} we call the “Larkin” scale, during the RG flow. By contrast, weaker nonanalyticities in $\Gamma_{k2}^{(11)}$ as well as nonanalyticities in higher-order cumulants (if no cusp has appeared in $\Gamma_{k2}^{(11)}$) can only appear at the fixed point, in the limit $k \rightarrow 0$, thereby preserving the dimensional-reduction property. Indeed, the derivation leading to Eq. (24) then remains valid.

C. Continuing the NP-FRG flow with broken SUSY

Another nontrivial question is then raised: if superrotational invariance (SUSY) is spontaneously broken, *how can one continue the RG flow for the effective average action?* The answer, which again should be verified in calculations, is as follows:

(i) Keeping in mind that superrotational invariance is explicitly broken in the presence of multiple copies (see section II) and that it is only recovered in the process of reducing the whole problem to a single-copy system, spontaneous breaking may then be described by rejecting any implicit assumption of analyticity of the field dependences in the latter process. One should therefore restrict the hierarchy of ERGE's to those equations for cumulants that are considered for generic (nonequal) field arguments, so that a putative nonanalytic dependence in these arguments can freely emerge along the RG flow.³⁴

(ii) One assumes that except for the superrotational invariance, all of the properties and symmetries of the effective average action Γ_k remain valid. In particular, this applies to the property of “Grassmannian ultralocality” of the random generating functional and to its consequences for the cumulants of the renormalized disorder. As discussed in the preceding paper, this property

is obeyed when taking the limit $\beta^{-1} = 0$ in the ERGE's for the cumulants, which leads to Eqs. (7,8) and their extensions to higher orders. It keeps track of the fact that only the ground state is considered for each copy. The validity of the property of “Grassmannian ultralocality” is therefore distinct from that of superrotational invariance, a distinction that cannot be made in the original Parisi-Sourlas formalism.

(iii) The cutoff functions satisfy the relation in Eq. (6) which contains as a special case the explicit superrotational invariance when $\Delta_k = \Delta_B Z_k$. In practice, the typical strength of the renormalized random field Δ_k is defined from $\Gamma_{k2}^{(11)}(q^2 = 0)$ evaluated for some specific uniform field configuration and the field renormalization constant Z_k is obtained from $\Gamma_{k1}^{(2)}(q^2 = 0)$ also evaluated for some uniform field configuration. Finally, the cusp in $\Gamma_{k2;x_1,x_2}^{(11)}(\phi_1, \phi_2)$, which is associated with spontaneous SUSY breaking, must be stable upon further evolution with the infrared scale k and must not generate “supercusps”. From the structure of the ERGE for $\Gamma_{k2;x_1,x_2}^{(11)}(\phi_1, \phi_2)$, it is expected that a “linear cusp”³² satisfies these requirements and provides a mechanism for dimensional reduction *if* it persists at the fixed point (see Appendix A).

IV. APPROXIMATION SCHEME

A. SUSY-compatible approximation scheme

Up to this point, many of the previous considerations remain plausible conjectures. We now provide an implementation of the NP-FRG that allows us to check their validity. The hierarchy of ERGE's of course cannot be solved exactly and the next step is to provide a SUSY-compatible nonperturbative approximation scheme. From our previous work,^{1,2,20,21} we know that an efficient scheme relies on a joint truncation in the *derivative expansion*, which approximates the long-distance behavior of the 1PI vertices,²⁸ and in the *expansion in cumulants of the renormalized disorder*. The truncations however must be combined in a way that does not explicitly break the supersymmetry, more precisely the superrotational invariance (SUSY). To implement this requirement, we use the WT identities.

As is clear from the discussion in the preceding section, a minimal truncation must at least include the second renormalized cumulant. When SUSY is not broken, the WT identity [see Eq. (22)] imposes that for a uniform field $\Gamma_{k2}^{(11)}(q^2; \phi, \phi)$ is given by the derivative of $\Gamma_{k1}^{(2)}(q^2; \phi)$ with respect to q^2 . (A similar relation is satisfied by the infrared cutoff functions.) Consider now the derivative expansion. In the lowest order (LPA for “local potential approximation”), the derivative of $\Gamma_{k1}^{(2)}$ is a field-independent constant, which would force $\Gamma_{k2}^{(11)}$ to be also field independent: this is clearly too crude

to describe the physics of the RFIM. The first order of the derivative expansion leads to $\Gamma_{k1}^{(2)}(q^2; \phi)$ of the form $U_k''(\phi) + Z_k(\phi)q^2$, with $U_k(\phi)$ the “effective average potential” (“Gibbs free energy” for a magnetic system) and $Z_k(\phi)$ a field renormalization function. When the WT identity is satisfied, this in turn requires that

$$\Gamma_{k2}^{(11)}(q^2; \phi, \phi) = \Delta_B Z_k(\phi), \quad (27)$$

which corresponds to a local approximation for the second cumulant. It is easy to generalize this reasoning by taking into account the whole set of WT identities for the proper vertices obtained from the cumulants: a SUSY-compatible approximation at order n consists of taking Γ_{k1} at the order n of the derivative expansion, Γ_{k2} at the order $n-1$ of the derivative expansion, ..., $\Gamma_{k(n+1)}$ in the local approximation, and all higher-order cumulants equal to zero.³⁶ This provides a scheme of successive approximations that in principle can be used to check the robustness of the results and if necessary improve them.

The minimal truncation that can already describe the long-distance physics of the RFIM and does not explicitly break SUSY is then the following:

$$\Gamma_{k1}[\phi] = \int_x \left[U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x)) (\partial_\mu \phi(x))^2 \right], \quad (28)$$

$$\Gamma_{k2}[\phi_1, \phi_2] = \int_x V_k(\phi_1(x), \phi_2(x)), \quad (29)$$

with the higher-order cumulants set to zero.³⁸

Inserted in the ERGE's for the cumulants, Eqs. (7) and (8), the above ansatz provides 3 coupled flow equations for the 1-copy effective average potential $U_k(\phi)$ (or its derivative) that describes the thermodynamics of the system, the field renormalization function $Z_k(\phi)$, and the 2-copy effective average potential $V_k(\phi_1, \phi_2)$ from which one obtains the second cumulant of the renormalized random field at zero momentum, $\Gamma_{k2}^{(11)}(q^2 = 0; \phi_1, \phi_2) \equiv \Delta_k(\phi_1, \phi_2) = V_k^{(11)}(\phi_1, \phi_2)$. SUSY is obeyed when $\Delta_k(\phi, \phi) = Z_k(\phi)$, which is easily satisfied at the UV scale $k = \Lambda$.

B. NP-FRG equations in a scaled form

In order to search for the fixed point that controls the critical behavior, the flow equations must be recast in a scaled form. This can be done by introducing appropriate scaling dimensions (see refs. [1,20] and paper III¹⁹). Near a zero-temperature fixed point,^{12,13} one has the following scaling dimensions:

$$Z_k \sim k^{-\eta}, \quad \Delta_k \sim k^{-(2\eta-\bar{\eta})}, \quad \phi_a \sim k^{\frac{1}{2}(d-4+\bar{\eta})}, \quad (30)$$

and a renormalized temperature is introduced as $T_k \sim \frac{k^2 Z_k}{\beta \Delta_k} \sim k^\theta / \beta$, with θ and $\bar{\eta}$ related through $\theta = 2 + \eta - \bar{\eta}$.

More precisely, we define running anomalous dimensions η_k and $\bar{\eta}_k$ as

$$\partial_t \log Z_k = -\eta_k, \quad (31)$$

$$\partial_t \log \Delta_k = -(2\eta_k - \bar{\eta}_k), \quad (32)$$

where Z_k and Δ_k have been introduced in Sec. IV-D. One also has

$$U_k \sim \frac{k^d}{T_k} \sim k^{d-\theta}, \quad V_k \sim \frac{k^d}{T_k^2} \sim k^{d-2\theta}. \quad (33)$$

The dimensionless counterparts of U_k, V_k, Δ_k, ϕ will be denoted by lower-case letters, u_k, v_k, δ_k, ϕ .

The resulting equations are

$$\begin{aligned} \partial_t u'_k(\varphi) = & -\frac{1}{2}(d-2\eta_k + \bar{\eta}_k)u'_k(\varphi) + \frac{1}{2}(d-4 + \bar{\eta}_k)\varphi u''_k(\varphi) \\ & + 2v_d \left\{ -\frac{d-2}{2}l_1^{(d-2)}(\varphi) u'''_k(\varphi) + [l_2^{(d+2)}(\varphi)z'_k(\varphi) + l_2^{(d)}(\varphi) \right. \\ & \times u'''_k(\varphi)] [z_k(\varphi) - \delta_{k,0}(\varphi)] + l_1^{(d)}(\varphi) \left[-\frac{d}{2}z'_k(\varphi) + \delta'_{k,0}(\varphi) \right] \\ & \left. + \frac{1}{2}(\eta_k - \bar{\eta}_k)[z'_k(\varphi)j_2^{(1,d+2)}(\varphi) + u'''_k(\varphi)j_2^{(1,d)}(\varphi)] \right\}, \end{aligned} \quad (34)$$

$$\begin{aligned} \partial_t z_k(\varphi) = & \eta_k z_k(\varphi) + \frac{1}{2}(d-4 + \bar{\eta}_k)\varphi z'_k(\varphi) - 2v_d \left\{ \frac{d-2}{2} \times \right. \\ & l_1^{(d-2)}(\varphi)z''_k(\varphi) - (d-2)l_2^{(d-2)}(\varphi)z'_k(\varphi)u'''_k(\varphi) + 2\frac{2d+1}{d} \times \\ & l_3^{(d+2)}(\varphi)z'_k(\varphi)^2[z_k(\varphi) - \delta_{k,0}(\varphi)] + 4l_3^{(d)}(\varphi)z'_k(\varphi)u'''_k(\varphi)[z_k(\varphi) \\ & - \delta_{k,0}(\varphi)] - l_2^{(d)}(\varphi) \left[\frac{2d+1}{2}z'_k(\varphi)^2 + z''_k(\varphi)(z_k(\varphi) - \delta_{k,0}(\varphi)) \right. \\ & \left. - 2z'_k(\varphi)\delta'_{k,0}(\varphi) \right] + m_4^{(d-2)}(\varphi)u'''_k(\varphi)^2 + \frac{2}{d}m_4^{(d)}(\varphi)u'''_k(\varphi) \times \\ & [(d+2)z'_k(\varphi) - 2\delta'_{k,0}(\varphi)] + \frac{1}{d}m_4^{(d+2)}(\varphi)z'_k(\varphi)[(d+4)z'_k(\varphi) - \\ & 4\delta'_{k,0}(\varphi)] - \frac{8}{d}[m_5^{(d+4)}(\varphi)z'_k(\varphi)^2 + 2m_5^{(d+2)}(\varphi)z'_k(\varphi)u'''_k(\varphi) + \\ & m_5^{(d)}(\varphi)u'''_k(\varphi)^2][z_k(\varphi) - \delta_{k,0}(\varphi)] - \frac{1}{2}(\eta_k - \bar{\eta}_k)[z''_k(\varphi) \times \\ & j_2^{(1,d)}(\varphi) - (4 + \frac{2}{d})z'_k(\varphi)^2 j_3^{(1,d+2)}(\varphi) - 4z'_k(\varphi)u'''_k(\varphi) \times \\ & j_3^{(1,d)}(\varphi) + \frac{2}{d}z'_k(\varphi)^2 h_4^{(d+2)}(\varphi) + \frac{4}{d}z'_k(\varphi)u'''_k(\varphi)h_4^{(d)}(\varphi) + \\ & \left. \frac{2}{d}u'''_k(\varphi)^2 h_4^{(d-2)}(\varphi) \right\}, \end{aligned} \quad (35)$$

$$\begin{aligned} \partial_t v_k(\varphi_1, \varphi_2) = & -(d-4 + 2\bar{\eta}_k - 2\eta_k)v_k(\varphi_1, \varphi_2) + \\ & \frac{1}{2}(d-4 + \bar{\eta}_k)(\varphi_1 \partial_{\varphi_1} + \varphi_2 \partial_{\varphi_2})v_k(\varphi_1, \varphi_2) - 2v_d \left\{ \frac{1}{2} \times \right. \\ & l_{1,1}^{(d)}(\varphi_1, \varphi_2)[\delta_k(\varphi_1, \varphi_2) - z_k(\varphi_1)]^2 + l_2^{(d)}(\varphi_1)v_k^{(20)}(\varphi_1, \varphi_2) \\ & \times [\delta_{k,0}(\varphi_1) - z_k(\varphi_1)] + \frac{d-2}{2}l_1^{(d-2)}(\varphi_1)v_k^{(20)}(\varphi_1, \varphi_2) + \\ & \frac{1}{2}m_{1,1}^{(d-2)}(\varphi_1, \varphi_2) - n_{1,1}^{(d-2)}(\varphi_1, \varphi_2)[\delta_k(\varphi_1, \varphi_2) - z_k(\varphi_1)] \\ & - \frac{1}{2}(\eta_k - \bar{\eta}_k)[\delta_k(\varphi_1, \varphi_2)j_{1,1}^{(1,d)}(\varphi_1, \varphi_2) + j_{1,1}^{(2,d)}(\varphi_1, \varphi_2) \\ & \left. + v_k^{(20)}(\varphi_1, \varphi_2)j_2^{(1,d)}(\varphi_1)] + perm(12) \right\}, \end{aligned} \quad (36)$$

where $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$, a prime denotes a derivative with respect to the field (when only one argument is present), $\delta_k(\varphi_1, \varphi_2) = v_k^{(11)}(\varphi_1, \varphi_2)$, $\delta_{k,0}(\varphi_1) = \delta_k(\varphi_1, \varphi_1)$, and $perm(12)$ denotes the terms obtained by permuting φ_1 and φ_2 .

The functions $l_q^{(d)}(\varphi)$, $l_{q_1, q_2}^{(d)}(\varphi_1, \varphi_2)$, $m_q^{(d)}(\varphi)$, $m_{q_1, q_2}^{(d)}(\varphi_1, \varphi_2)$, $n_{q_1, q_2}^{(d)}(\varphi_1, \varphi_2)$, $h_q^{(d)}(\varphi)$, $j_{q_1}^{(a,d)}(\varphi)$ and $j_{q_1, q_2}^{(a,d)}(\varphi_1, \varphi_2)$ appearing in the above flow equations are “dimensionless threshold functions” which are defined from the infrared cutoff functions that satisfy³⁵

$$\hat{R}_k(q^2) = Z_k k^2 s(q^2/k^2) \quad (37)$$

and

$$\tilde{R}_k(q^2) = -\Delta_k s'(q^2/k^2), \quad (38)$$

with the prime again denoting a derivative. The dependence of the dimensionless threshold functions on the dimensionless fields comes through $u''_k(\varphi)$ and $z_k(\varphi)$. Their definitions are given in Appendix B. The properties of these threshold functions have been extensively discussed.^{28,39,40} They decay rapidly when their arguments $u''_k(\varphi_a)$ become large, which, since $u''_k(\varphi_a) = U''_k(\varphi_a)/(Z_k k^2)$ is the square of a renormalized mass, ensures that only modes with a mass smaller than k contribute to the flow in Eqs. (34-36). These threshold functions essentially encode the nonperturbative effects beyond the standard one-loop approximation.^{28,39,40}

Note that $u_k(\varphi)$ and $z_k(\varphi)$ are even functions of φ and that, due to the Z_2 symmetry and the permutational symmetry of the cumulants in their field arguments, $v_k(\varphi_1, \varphi_2)$ satisfies $v_k(\varphi_1, \varphi_2) = v_k(-\varphi_1, -\varphi_2) = v_k(\varphi_2, \varphi_1) = v_k(-\varphi_2, -\varphi_1)$, and similarly for $\delta_k(\varphi_1, \varphi_2)$. Finally, the initial conditions for Eqs. (31,32,34,35,36) at the UV scale Λ ($t = 0$) are obtained from the bare action in Eq. (4) for $\beta = 0$ and $\Phi_a = \varphi_a$, i.e.,

$$z_\Lambda(\varphi) = 1 \quad (39)$$

$$u_\Lambda(\varphi) = \frac{\tau}{2}\varphi^2 + \frac{u}{4!}\varphi^4 = \frac{u}{4!}(\varphi^2 - \kappa_\Lambda)^2, \quad (40)$$

the last equality being valid in the (relevant) region where the bare potential has a nontrivial minimum,²⁸ after a

trivial rescaling of the fields so that the explicit dependence on Δ_B disappears, one also has

$$v_\Lambda(\varphi_1, \varphi_2) = \varphi_1 \varphi_2, \quad (41)$$

which implies $\delta_\Lambda(\varphi_1, \varphi_2) = 1$. This initial condition trivially satisfies $\delta_\Lambda(\varphi, \varphi) = z_\Lambda(\varphi)$ which ensures that invariance under superrotations is obeyed at the beginning of the flow.

C. Properties of the flow equations

An important property of the present theory occurs in the limit $\varphi_2 \rightarrow \varphi_1$ when $\delta_k(\varphi_1, \varphi_2)$ is regular enough. For convenience, we introduce the variables $\varphi = (\varphi_1 + \varphi_2)/2$ and $y = (\varphi_1 - \varphi_2)/2$. The property can be stated as follows: If the behavior of δ_k is regular enough when $y \rightarrow 0$, *i.e.*, is such that

$$\delta_k(\varphi, y) = \delta_{k,0}(\varphi) + \frac{1}{2}\delta_{k,2}(\varphi)y^2 + y^2 o(y) \quad (42)$$

with $o(y) \rightarrow 0$ when $y \rightarrow 0$, then the flow of $\delta_{k,0}(\varphi)$ coincides with that of $z_k(\varphi)$. This is precisely the WT relation in Eq. (22) when evaluated at zero momentum; indeed, in dimensionless form, this equation is just $\delta_{k,0}(\varphi) = z_k(\varphi)$. Within the present ansatz for the effective average action, dimensional reduction then follows.

To prove the equality of the two flow equations, we first derive Eq. (36) with respect to φ_1 and φ_2 to obtain the RG equation for δ_k , change variables to φ, y and take the limit $y = 0$ by assuming the above small y behavior [Eq. (42)]. The derivation makes use of the properties of the dimensionless threshold functions that are summarized in Appendix B. This leads to

$$\begin{aligned} \partial_t \delta_{k,0}(\varphi) = & -(\bar{\eta}_k - 2\eta_k)\delta_{k,0}(\varphi) + \frac{1}{2}(d-4+\bar{\eta}_k)\varphi\delta'_{k,0}(\varphi) \\ & - 2v_d \left\{ \frac{d-2}{2} \left[l_1^{(d-2)}(\varphi)\delta''_{k,0}(\varphi) - 2l_2^{(d-2)}(\varphi)u_k'''(\varphi)\delta'_{k,0}(\varphi) \right] \right. \\ & - \frac{2d-3}{2}l_2^{(d)}(\varphi)\delta'_{k,0}(\varphi)^2 + m_4^{(d-2)}(\varphi)u_k'''(\varphi)^2 + 2m_4^{(d)}(\varphi) \times \\ & \left. u_k'''(\varphi)\delta'_{k,0}(\varphi) + m_4^{(d+2)}(\varphi)\delta'_{k,0}(\varphi)^2 + L_k[\varphi; \delta_{k,0} - z_k] \right\}, \end{aligned} \quad (43)$$

where $L_k[\varphi; \delta_{k,0} - z_k]$ is a polynomial of degree two in $[\delta_{k,0}(\varphi) - z_k(\varphi)]$ and its derivatives, with coefficients that are function of φ and such that $L_k[\varphi; 0] \equiv 0$. Subtracting from the above flow equation the one for z_k in Eq. (35) leads to an equation for the difference $\delta_{k,0} - z_k$ in the form

$$\partial_t [\delta_{k,0}(\varphi) - z_k(\varphi)] = M_k[\varphi; \delta_{k,0} - z_k] - (\bar{\eta}_k - \eta_k)z_k(\varphi), \quad (44)$$

where $M_k[\varphi; \delta_{k,0} - z_k]$ is another polynomial of degree two in $[\delta_{k,0}(\varphi) - z_k(\varphi)]$ and its derivatives such that $M_k[\varphi; 0] \equiv 0$. [Note that due to their definitions, $\bar{\eta}_k$ and η_k are equal as soon as $\delta_{k,0}(\varphi) = z_k(\varphi)$.] Since the initial

condition satisfies $\delta_{k=\Lambda}(\varphi_1, \varphi_2) \equiv 1$ and $z_{k=\Lambda}(\varphi) \equiv 1$, the only solution to Eq. (44) is $\delta_{k,0}(\varphi) - z_k(\varphi) = 0$ at all scales, which provides the desired demonstration. By inserting this result in Eq. (34), one obtains

$$\begin{aligned} \partial_t u_k'(\varphi) = & -\frac{1}{2}(d-\eta_k)u_k'(\varphi) + \frac{1}{2}(d-4+\eta_k)\varphi u_k''(\varphi) \\ & - (d-2)v_d \left\{ l_1^{(d-2)}(\varphi)u_k'''(\varphi) + l_1^{(d)}(\varphi)z_k'(\varphi) \right\}, \end{aligned} \quad (45)$$

which coincides with the flow equation for $u_k'(\varphi)$ in the φ^4 theory without random field in dimension $d-2$ (use the relation $(d-2)v_d = (2\pi)^{-1}v_{d-2}$ and compare with Eq. (4.33) of Ref. [28]). The same exercise can be repeated for the flow of $z_k(\varphi)$, thereby completing the proof of dimensional reduction in this nonperturbative approximation.

On the other hand, a spontaneous breaking of SUSY and of the associated WT identity occurs whenever $\delta_{k,2}(\varphi)$ diverges and δ_k has a cusp-like singularity in the form

$$\delta_k(\varphi_1, \varphi_2) = \delta_{k,0}(\varphi) + \delta_{k,1}(\varphi)|\varphi_1 - \varphi_2| + \dots \quad (46)$$

as $\varphi_2 \rightarrow \varphi_1 \rightarrow \varphi$. We expect that such a cusp appears at a finite ‘‘Larkin’’ scale k_L through the formation of a boundary layer as $k \rightarrow k_L$ and $y \rightarrow 0$ in the form

$$\partial^2 y \delta_k(\varphi, y) \sim \frac{1}{[k - k_L(\varphi)]^{2\varpi} + y^2}, \quad (47)$$

with ϖ an (*a priori* unknown) exponent, as can be checked by inserting in the flow equations. The boundary layer provides the appropriate initial conditions to further continue the flow below k_L with a cuspy function.

Finally, it is worth comparing the above NP-FRG equations with those we have previously derived without the superfield formalism^{1,2} in a minimal approximation based on the same ansatz as in Eqs. (28,29). The two main differences are that:

(i) The present equations are directly considered at zero temperature (and zero auxiliary temperature) and subdominant terms proportional to T_k do not appear in the beta functionals.

(ii) More importantly, since from the previous formalism we had no insight into the requirements for avoiding an *explicit* breaking of the underlying superrotational invariance at the single-copy level, we had for simplicity set the infrared cutoff function \tilde{R}_k to zero. As a result, the WT identity associated with superrotational invariance was violated at *all* scales. The dimensional-reduction property could therefore not be exactly recovered when it was expected to be valid.^{1,2}

D. Numerical resolution

We are interested in the long-distance physics of the RFIM near its critical point and in the associated (once

unstable) fixed point. Such information can be obtained from the solution of the above coupled FRG flow equations with appropriate boundary conditions. We have carried out the three following types of resolution.

(1) We have investigated the location of the breakdown of SUSY associated with the appearance of a cusp in $\delta_k(\varphi_1, \varphi_2)$. To this end, we have solved the set of coupled flow equations which is obtained when δ_k has a sufficiently regular (no cusp) behavior as $\varphi_1 \rightarrow \varphi_2$, *i.e.* $y \rightarrow 0$: see Eq. (42). As already shown, the flow of $\delta_{k,0}(\varphi)$ then coincides with that of $z_k(\varphi)$, and the NP-FRG equations for $u'_k(\varphi)$ and $z_k(\varphi)$ form a closed set. In addition, we have solved the equation for the second derivative of δ_k with respect to $(\varphi_1 - \varphi_2)$, $\delta_{k,2}(\varphi)$, which depends on $u'_k(\varphi)$ and $z_k(\varphi)$. A divergence in $\delta_{k,2}(\varphi)$ signals the appearance of a cusp in $\delta_k(\varphi_1, \varphi_2)$.

The numerical task of solving this system of three coupled partial differential equation is reasonable. We discretize the problem both in the time and field directions. We use finite differences to evaluate the field derivatives that appear in the beta functions and use an explicit scheme to find the (RG) “time” evolution of the functions.

(2) When a cusp is present, solving in their full glory the set of coupled partial differential equations for functions of up to two field arguments is a hard task. We have discretized the field variables, using a 2-dimensional grid to solve the equation for $\delta_k(\varphi_1, \varphi_2)$. For convenience, we have switched variables from φ_1, φ_2 to $X = (\varphi_1 + \varphi_2)/2$ and $y = (\varphi_1 - \varphi_2)/2$ and restricted the grid to a trapezoidal region with $X \geq 0$ and $y \geq 0$. Observe that the flow of $\delta_k(\varphi_1, \varphi_2)$ depends on $\delta_k(\varphi_1, \varphi_1)$ and $\delta_k(\varphi_2, \varphi_2)$ and their derivatives. In this sense, the flow equation for δ_k is nonlocal since its evolution depends not only on its property in a neighborhood of (φ_1, φ_2) but also on its values at points faraway in the (φ_1, φ_2) plane. It is therefore important to choose the trapezoidal grid such that, for all point in the grid, the information necessary for computing the evolution of the function at this point is inside the trapezoid. In the larger simulations we had 60 points in the X direction and 50 in the y direction.

We have solved the set of coupled equations with two kinds of *modus operandi*:

(2-i) First, for selected values of the dimension d we have searched the “cuspy” fixed point by dichotomy, starting from “cuspy” initial conditions ($\delta_{k=\Lambda,1}(x) \neq 0$). Depending on the dimension and the regulator parameters, there appears from time to time numerical instabilities. Such instabilities showed up for instance in the (physically not very interesting) large-field region, where the threshold functions are very small. To keep the latter under control, we have therefore used slightly modified flow equations: this is discussed in Appendix C.

(2-ii) Secondly, once a proper “cuspy” fixed point has been identified in some given d , we have followed this fixed point by continuously decreasing or increasing d . This was done by looking directly for the roots of the beta functions by means of the Hybrid algorithm, using

as an initial guess the fixed point found at the previous iteration. We have studied the cuspy fixed point down to $d \lesssim 3$; it is indeed difficult to go down to lower d because the anomalous dimensions η and $\bar{\eta}$ become large in low dimensions: $\bar{\eta} \simeq 4 - d$ (see also Ref. [41]).

(3) When approaching the critical dimension d_{DR} for both SUSY and dimensional-reduction breaking from below, the procedures (2-i) and (2-ii) above become inefficient. We have used instead an expansion of the function $\delta_k(\varphi_1 = x + y, \varphi_2 = x - y)$ in $|y|$ with, in practice,

$$\delta_k(x+y, x-y) = \delta_{k,0}(x) + |y|\delta_{k,1}(x) + \frac{y^2}{2}\delta_{k,2}(x) + O(|y|^3). \quad (48)$$

Note that the expansion cannot be applied in the close vicinity of d_{DR} . Indeed, one then expects the approach $d \rightarrow d_{DR}^-$ to proceed nonuniformly in y through a boundary layer in $y^2/(d_{DR} - d)$. It is easy to check that a similar boundary-layer phenomenon is found in the more easily handled 1-loop equation for the RFO(N)M near $d = 4$,⁴² around the critical value $N_{DR} = 18$, *i.e.* when $N \rightarrow 18^-$.

To compute the anomalous dimensions, we have defined the amplitudes Z_k and Δ_k appearing in Eqs. (31,32) by choosing the specific configuration of the fields as equal to zero. This amounts to imposing

$$\partial_t z_k(\varphi = 0) = 0, \quad (49)$$

$$\partial_t \delta_{k,0}(\varphi) = \partial_t \delta_k(\varphi, \varphi) = 0, \quad (50)$$

which fixes η_k and $\bar{\eta}_k$.

In addition to the anomalous dimensions η and $\bar{\eta}$ (obtained as the fixed-point values η_* and $\bar{\eta}_*$), we have computed the critical exponent ν that controls the escape of the RG flow away from the fixed point along the unstable (relevant) direction. This was done by diagonalizing the flow equations around the fixed point and extracting the negative eigenvalue which identifies with $-1/\nu$.

Finally, we have chosen a dimensionless cutoff function $s(q^2/k^2)$ [see Eqs. (37) and (38)] of the form

$$s(y) = (a + by + cy^2) e^{-y}, \quad (51)$$

where $y = q^2/k^2$ [not to be confused with $(\varphi_2 - \varphi_1)/2$]. This function satisfies the requirements $s(y \rightarrow \infty) \rightarrow 0$ and $s(y \rightarrow 0) = \text{cst} > 0$. It is a modification of the function used in Refs. [27,28]. The parameters a , b and c can be optimized via stability considerations (see Refs. [43,44,45]). They are varied to find a region of values for which the computed critical exponents are stable under changes of the parameters. For more details, see Appendix C.

V. RESULTS AND DISCUSSION

A. Results

Our main result is a numerical confirmation of the scenario put forward above: the NP-FRG equations allow-

ing us to continuously follow the critical behavior and the associated fixed point as a function of dimension d , we find that there is a critical dimension d_{DR} separating a region $d > d_{DR}$ where SUSY is valid all along the RG flow (except possibly right at the fixed point) and dimensional reduction applies from a region $d < d_{DR}$ where SUSY is broken at a finite IR scale along the RG flow and a breakdown of dimensional reduction takes place. By using the procedure (1) detailed in the above subsection, we have numerically located this critical dimension as $d_{DR} \simeq 5.1 \pm 0.1$ (the precise value has a residual dependence on the chosen cutoff function). Note that the value $d_{DR} \simeq 5.1$ obtained here is consistent with the value found in our previous, and somewhat cruder, NP-FRG approach of the RFO(N)M when extrapolating the transition line $d_{DR}(N)$ down to $N = 1$ (see Figure 4 of paper II²).

For initial conditions of the RG flow at or near the critical point, the second derivative $\delta_{k,2}(\varphi) = \partial_y^2 \delta_k(\varphi + y, \varphi - y)|_{y=0}$ blows up at a finite RG “time” $t_L = \log(k_L/\Lambda)$ for $d < d_{DR}$, whereas it stays finite up to the fixed point for $d > d_{DR}$.⁴⁶ We illustrate the difference of behavior between these two cases in Fig. 1 for a field configuration $\varphi = 0$. The divergence of the full function $\delta_{k,2}(\varphi)$

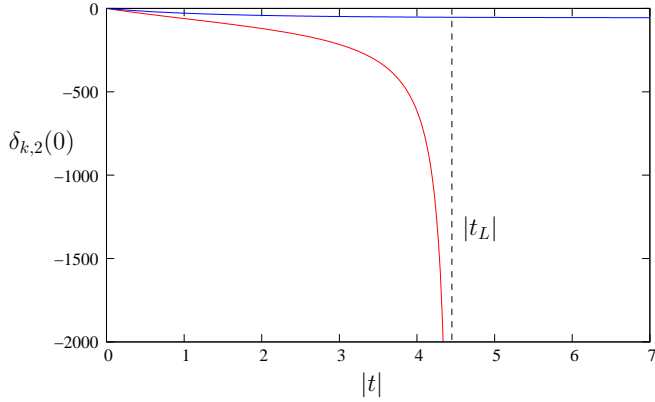


FIG. 1: NP-FRG flow of $\delta_{k,2}(0)$ in the regime where SUSY is valid. The initial conditions at $k = \Lambda$ (i.e., $t = 0$) for $u'_k(\rho)$ and $z_k(\rho) = \delta_{k,0}(\rho)$, with $\rho = \varphi^2/2$, are taken at the fixed-point solution, $u'_*(\rho)$ and $z_*(\rho)$ [$\partial_t u'_k(\rho)|_* = \partial_t z_k(\rho)|_* = 0$], and those for $\delta_{k,0}(\rho)$ and $\delta_{k,2}(\rho)$ are chosen as $\delta_{k=\Lambda,0}(\rho) = z_*(\rho)$ and $\delta_{k=\Lambda,2}(\rho) = 0$. The upper (color online blue) curve corresponds to $d = 5.2 > d_{DR}$ and one observes that $\delta_{k,2}(0)$ tends to a finite fixed-point value. The lower (color online red) curve corresponds to $d = 5 < d_{DR}$ shows a divergence at a finite RG “time” t_L .

when $d < d_{DR}$ is shown in Fig. 2. (Due to the Z_2 symmetry, it is more convenient to represent the functions in terms of $\rho = \varphi^2/2$.) We find, as can be anticipated from an analysis of the NP-FRG equations for $u'_k(\varphi)$, $\delta_{k,0}(\varphi)$ and $\delta_{k,2}(\varphi)$, that the latter stays finite at $d = d_{DR}$ and that its fixed-point value for $d \rightarrow d_{DR}^+$ behaves as $\delta_{*,2}(\varphi)|_d - \delta_{*,2}(\varphi)|_{d_{DR}} \propto \sqrt{d - d_{DR}}$, as seen in Fig. 3.

When considering dimensions smaller than d_{DR} , one must study the full dependence of the function $\delta_k(\rho =$

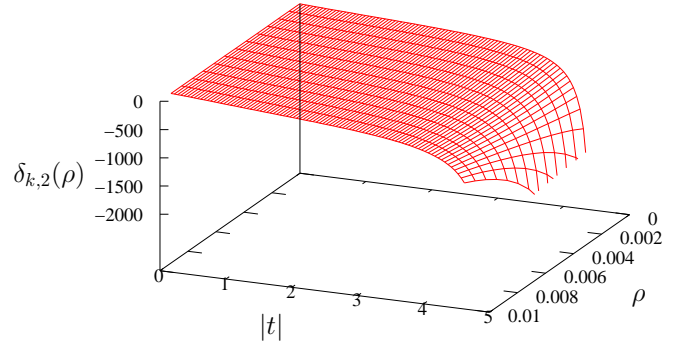


FIG. 2: NP-FRG flow of $\delta_{k,2}(\rho) = \partial_y^2 \delta_k(\rho, y)|_{y=0}$ for $d = 5 < d_{DR}$. The initial conditions at $k = \Lambda$ (i.e., $t = 0$) for $u'_k(\rho)$ and $z_k(\rho)$ are taken at the fixed-point solution, $u'_*(\rho)$ and $z_*(\rho)$, and those for $\delta_{k,0}(\rho)$ and $\delta_{k,2}(\rho)$ are chosen as $\delta_{k=\Lambda,0}(\rho) = z_*(\rho)$ and $\delta_{k=\Lambda,2}(\rho) = 0$. One observes that the divergence first takes place for small values of ρ .

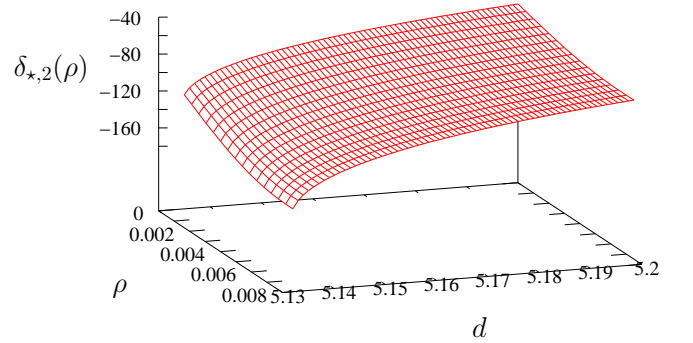


FIG. 3: Fixed-point solution $\delta_{*,2}(\rho)$ for dimensions $d > d_{DR}$. Note the square-root behavior $\sqrt{d - d_{DR}}$ as a function of dimension.

$\varphi^2/2, y)$. We show in Fig. 4 the evolution of this function for $\rho = 0$. Starting from a constant function, one can clearly see that a linear cusp in y appears in a finite RG time $|t_L|$, close to 0.7 for the case shown.

The appearance of a cusp along the flow leads to a breakdown of the superrotational invariance and of the associated WT identities. This is illustrated for the $d = 3$ fixed point in Fig 5: there, $z_*(\rho) \neq \delta_{*,0}(\rho) \equiv \delta_*(\rho, 0)$, which implies a breaking of the WT identity in Eq. (27). The different asymptotic behaviors at large ρ are easily deduced from Eqs. (35,36), from which we show that $z_*(\rho) \sim \rho^{-\eta/(d-4+\bar{\eta})}$ and $\delta_*(\rho, 0) \sim \rho^{-(2\eta-\bar{\eta})/(d-4+\bar{\eta})}$. In $d = 3$, $2\eta - \bar{\eta}$ is very small, which implies that the function $\delta_*(\rho, 0)$ decreases slowly to 0.

We also display in Figs. 6 and 7 the fixed-point solutions $u'_*(\rho)$ and $z_*(\rho)$ for different dimensions. One can see that the lower the dimension the steeper the curves. This means that for a numerical study of the critical properties in low dimensions, we need to discretize the field dependence in the NP-FRG equations with a small mesh, which entails a large number of points. The numerical integration is therefore more difficult and even be-

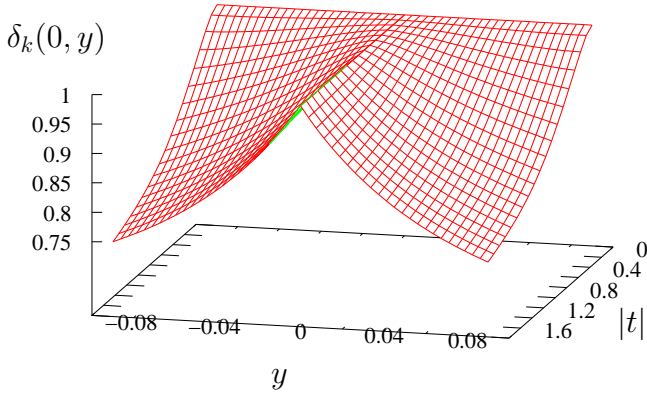


FIG. 4: NP-FRG flow of $\delta_k(0, y)$ for $d = 4 < d_{DR}$. The initial conditions at $k = \Lambda$ (i.e., $t = 0$) for $u'_k(\rho)$ and $z_k(\rho)$ are taken at the fixed-point solution, $u'_*(\rho)$ and $z_*(\rho)$, and that for δ_k is chosen as $\delta_{k=\Lambda}(\rho, y) = 1$. One observes that a linear cusp appears at a finite RG time. By construction $\delta_k(0, 0) = 1$ along the flow.

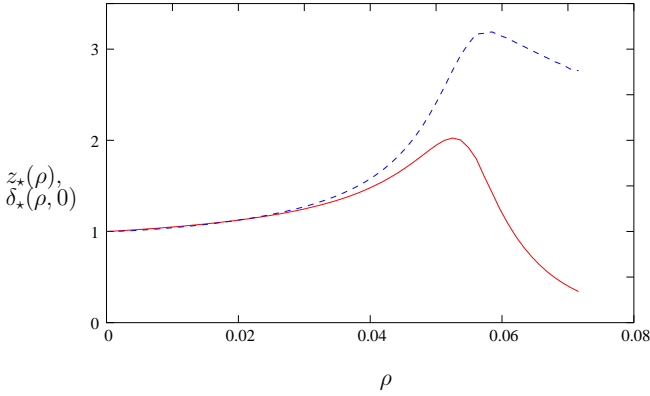


FIG. 5: Fixed-point solution in $d = 3$ for $z_*(\rho)$ (solid line) and $\delta_*(\rho, 0)$ (dashed line). The two functions differ for a large enough field (by construction, they coincide at $\rho = 0$).

comes intractable in practice. Finally, we show in Fig. 8 the full dependence of the fixed-point solution $\delta_*(\rho, y)$.

We now turn to the results concerning the critical exponents. We begin with the anomalous dimensions η and $\bar{\eta}$ which are determined at the (critical) fixed point. (We recall that η characterizes the spatial decay of the “connected” pair correlation function and $\bar{\eta}$ that of the “disconnected” pair correlation function at criticality.) Their dependence on the spatial dimension d is shown in Fig. 9. Above $d_{DR} \simeq 5.1$, we find that η and $\bar{\eta}$ rigorously coincide, $\bar{\eta} = \eta$, which is the signature of dimensional reduction. (Their value is moreover equal to that found in the same order of the derivative expansion for the pure Ising model in dimension $d - 2$.) Below d_{DR} , the two anomalous dimensions bifurcate and $\bar{\eta} \neq \eta$. Note that the predicted values satisfy the required bounds, i.e., $2\eta \geq \bar{\eta} \geq \eta$, $\eta \geq (4 - d)/2$, $\bar{\eta} \geq 4 - d$.

We give in Table I our estimates for η and $\bar{\eta}$ in $d = 3$ and $d = 4$, where a comparison is possible with numerical

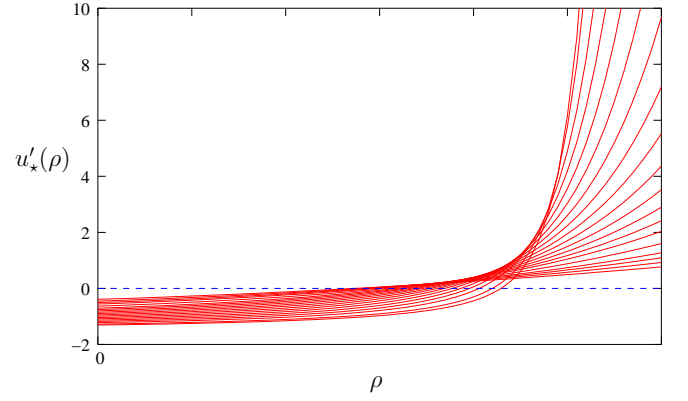


FIG. 6: Fixed-point solution $u'_*(\rho)$ for dimensions ranging from 3 (steepest) to 4.8 (smoother curve). The variable ρ has been rescaled by a factor e^{-d} so that all curves can be represented on the same plot.

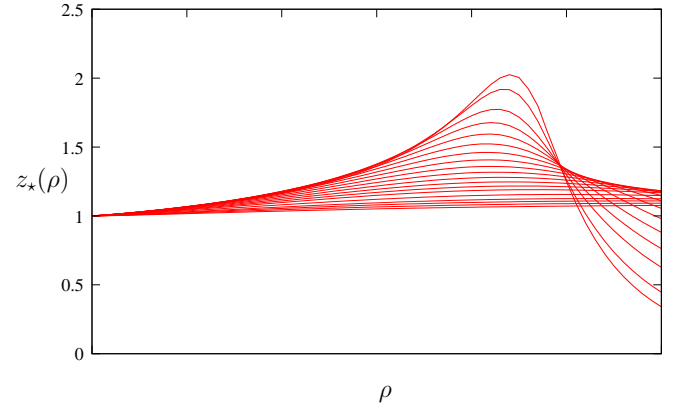


FIG. 7: Fixed-point solution $z_*(\rho)$ for dimensions ranging from 3 (steepest) to 4.8 (smoother curve). The variable ρ has been rescaled by a factor e^{-d} so that all curves can be represented on the same plot.

determinations from computer studies (Monte Carlo calculations and $T = 0$ ground-state determinations). One can see that the agreement is good or very good. (Real-space RG studies in $d = 3^{61-65}$ provide results in the same range, with an anomalous dimension η between 0.51 to 0.56 and an exponent θ between 1.50 and 1.56.) As is well known from the study of simpler models such as the pure $O(N)$ model, the way to further improve the accuracy of the exponents would be to consider higher orders of the approximation scheme presented in section IV-A.

We have not been able to perform our NP-FRG calculation down to the lower critical dimension $d = 2$, where the values $\eta = 1$ and $\bar{\eta} = 2$ are exactly known. The numerical resolution of the flow equations become extremely arduous in low dimension where the anomalous dimensions become large and approach their lower bound, $(4 - d)/2$ and $4 - d$ respectively. In addition, one also encounters numerical difficulties as one approaches the critical dimension d_{DR} from below. Indeed, as men-

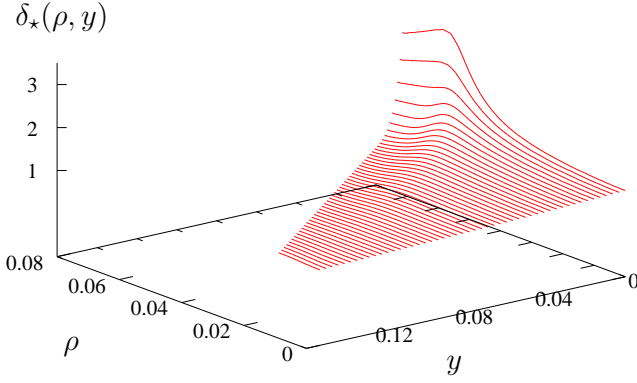


FIG. 8: Fixed-point solution $\delta_*(\rho, y)$ in $d = 3$. Note that the point $(\rho = 0, y = 0)$ corresponds to the lower right corner. The function is even in y (with a cusp around $y = 0$) and the part corresponding to $y < 0$ is not shown.

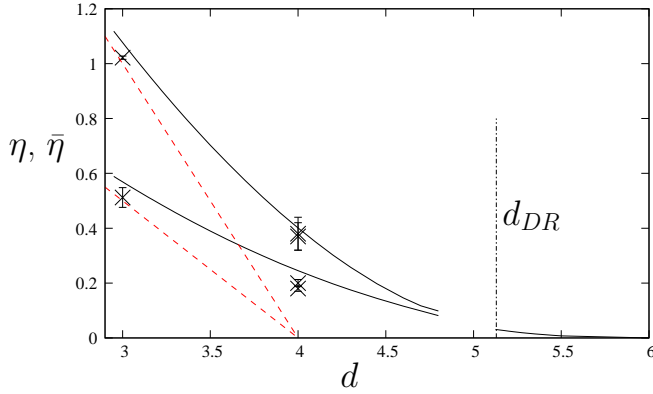


FIG. 9: Anomalous dimensions η and $\bar{\eta}$ as functions of the spatial dimension d . The dashed lines are lower bounds for the anomalous dimensions $[(4 - d)/2$ and $4 - d$ for η and $\bar{\eta}$ respectively] and the crosses correspond to predictions from computer simulations and ground-state determinations.

tioned in section IV-D, it is anticipated that the limit $d \rightarrow d_{DR}^-$, $y \rightarrow 0$ is nonuniform with a boundary layer in $y^2/(d_{DR} - d)$. More work will be needed to solve this boundary-layer problem numerically.

Fig. 9 also provides evidence that the claim according to which the two exponents η and $\bar{\eta}$ are related by a fixed ratio, $\bar{\eta} = 2\eta$,^{22–24,66} cannot be right in general. It is true that the relation $\bar{\eta} = 2\eta$ is exact in $d = 2$ (and according to a phenomenological RG,⁶⁷ also at first order in $d = 2 + \epsilon$) and is very closely obeyed by the numerical estimates in $d = 3$ and $d = 4$ (see Table I). However, the latter type of “numerical evidence” can always be challenged, and it is actually impossible to reach a definite conclusion on the sole basis of numerical results in selected dimensions ($d = 3, 4, 5$). On the other hand, the overall and continuous dependence on spatial dimension that we provide through the NP-FRG leads to a different and firmer type of answer: since one goes from $\bar{\eta} = \eta$ for $d \geq d_{DR}$ to $\bar{\eta} > \eta$ above, the relation $\bar{\eta} = 2\eta$ cannot be

always valid.

Finally, we also display our results in $d = 3$ and $d = 4$ for the correlation length exponent ν , which is associated with the relevant direction around the critical fixed point, in Table I. The agreement with the available data from computer studies is excellent in $d = 4$ and fair in $d = 3$. (Note that in the real-space RG studies in $d = 3$ ^{61–65}, the exponent ν is in general too large, with values ranging from 1.39 to 2.25.) All the other critical exponents are obtained in the NP-FRG through the expected relations: $\gamma = (2 - \eta)\nu$, $\bar{\gamma} = (4 - \bar{\eta})\nu$, $\delta = (d + 4 - \bar{\eta})/(d - 4 + \bar{\eta})$, etc.⁶⁸

The validity of the theoretical description is clearly confirmed by both the overall consistency and the quantitative accuracy of the predictions in $d = 3$ and $d = 4$.

B. Role of the auxiliary temperature and correction to “Grassmannian ultralocality”

To assess the effect of a nonzero auxiliary temperature and of a deviation from “Grassmannian ultralocality”, we have considered the simplest “non-ultralocal” contribution to the cumulants introduced in section VII-B of paper III. Combined with the above truncation of the “ultralocal” contributions of effective average action [Eqs. (28,29)], this amounts to expressing the first cumulant as (see Eq. (136) of paper III)

$$\Gamma_{k1}[\Phi_1] = \int_{\underline{\theta}_1} \left(\Gamma_{k1}[\Phi_1(\underline{\theta}_1)] + \frac{1}{2\beta}(1 + \beta\bar{\theta}_1\theta_1) \times \int_x Y_k(\Phi_1(x, \underline{\theta}_1)) \partial_{\theta_1} \Phi_1(x, \underline{\theta}_1) \partial_{\bar{\theta}_1} \Phi_1(x, \underline{\theta}_1) \right), \quad (52)$$

with $\Gamma_{k1}[\phi]$ given in Eq. (28), whereas the second-order cumulant is taken as purely “ultralocal” and given in

d	η (this work)	η (ground state)	η (Monte Carlo)
3	0.57(2)	0.51 ^a	0.5-0.6 ^{f,g,h}
4	0.24(1)	0.18 ^b , 0.20 ^c
d	$\bar{\eta}$ (this work)	$\bar{\eta}$ (ground state)	$\bar{\eta}$ (Monte Carlo)
3	1.08(2)	1.02 ^a , 1.06 ^d , 1.1 ^e	1.0-1.04 ^{f,g,h}
4	0.40(?)	0.37 ^b , 0.38 ^c	...
d	ν (this work)	ν (ground state)	ν (Monte Carlo)
3	1.08(2)	1.0 ^e , 1.1 ⁱ , 1.19 ^d , 1.22 ^j , 1.25 ^k , 1.36 ^l , 1.37 ^a	1.0 ^m , 1.1 ^g , 1.31 ⁿ
4	0.81(3)	0.78 ^b , 0.82 ^c	

TABLE I: Comparison between the anomalous dimensions obtained in the present work and in computer studies (a: [47]; b: [48]; c: [49]; d: [50]; e: [51]; f: [52]; g: [53]; h: [54]; i: [55]; j: [56]; k: [57]; l: [58]; m: [59]; n: [60]). The error bars on the results from simulations and ground-state computations can be found in the original papers. The values of the temperature exponent $\theta = 2 + \eta - \bar{\eta}$ that we obtain are 1.49 in $d = 3$ and 1.84 in $d = 4$, in excellent agreement with the other studies.

Eq. (29). The FRG flow equation for the function $Y_k(\phi)$ is directly obtained from Eq. (139) of paper III. In a dimensionless form, it reads

$$\begin{aligned} \partial_t y_k(\varphi) = & (2\eta_k - \bar{\eta}_k)y_k(\varphi) + \frac{d-4+\bar{\eta}_k}{2}\varphi y'_k(\varphi) + \\ & v_d \left(2y''_k(\varphi) - \frac{y'_k(\varphi)^2}{y_k(\varphi)} \right) \left\{ (d-2)l_1^{(d-2)}(\varphi) + 2[\delta_{k,0}(\varphi) \right. \\ & \left. - z_k(\varphi)]l_2^{(d)}(\varphi) + (\bar{\eta}_k - \eta_k)j_2^{(1,d)}(\varphi) \right\} + O(T_k), \end{aligned} \quad (53)$$

where the threshold functions are defined in Appendix B. In deriving the above expression from Eq. (139) of the preceding paper, we have used the dimensionless quantity y_k defined through $Y_k(\phi) = (Z_k k^2 / T_k) y_k(\varphi)$ and the fact that as $k \rightarrow 0$,

$$\begin{aligned} \hat{Q}_k(q^2, \phi) - \hat{P}_k(q^2, \phi) &= \frac{1}{\hat{P}_k(q^2, \phi)^{-1} + Y_k(\phi)} \\ &\sim \frac{1}{Y_k(\phi)} \sim \frac{k^{2\eta-\bar{\eta}}}{y_k(\varphi)}. \end{aligned} \quad (54)$$

The above equation is valid provided that \hat{P}_k^{-1} which asymptotically goes as $k^{2-\eta}$ is subdominant with respect to Y_k . As discuss just below, this is indeed true, although y_k goes to zero at the fixed point.

The flow of the “ultralocal” quantities in the limit $\beta^{-1} = 0$ has already been solved and we can introduce the corresponding solution in Eq. (53). We have solved the latter equation and found that $y_k(\varphi)$ goes to zero as one approaches the fixed point, in such a way that the dimensionful quantity $Y_k(\varphi)$ goes to a constant Y_* : this is illustrated in Fig. 10 for $d = 3$.

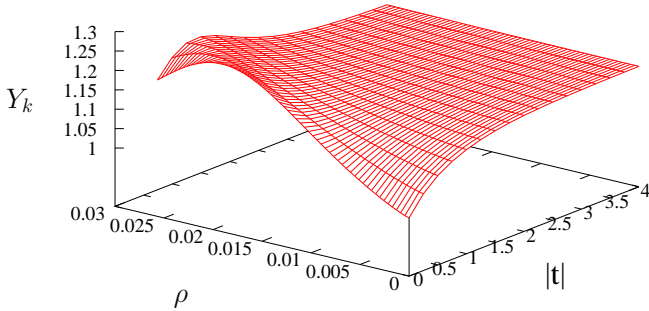


FIG. 10: Flow of the (dimensionful) function Y_k of the dimensionless field with $\rho = \varphi^2/2$. After a transient regime, the function tends to a constant.

In addition, we can investigate the effect of the “non-ultralocal” contribution to the flow of the “ultralocal” quantities when the auxiliary temperature β^{-1} is different from zero. This can be done by considering the ERGE’s for the “ultralocal” components of the first and the second cumulants given in Eqs. (134) and (135) of paper III. From the above results (in particular, the fact that Y_k goes to a finite constant when $k \rightarrow 0$), we know

that $\hat{Q}_k(q^2, \phi)$ is asymptotically equal to $\hat{P}_k(q^2, \phi)$: see Eq. (54) and discussion below. The flow of the dimensionless first cumulant therefore follows Eq. (151) of paper III and the contribution of the “non-ultralocal” piece goes as T_k times a well-behaved function(al), with $T_k \sim k^\theta/\beta$ and $\theta = 2 + \eta - \bar{\eta} > 0$.

The flow of the second cumulant is also of the form of Eq. (151) in the preceding paper, but the dimensionless beta function due to the “non-ultralocal” piece is now potentially singular when a cusp appears in the limit of zero (auxiliary) temperature of the second cumulant. More specifically, one finds that

$$\begin{aligned} \partial_t \delta_k(\varphi_1, \varphi_2) &= \beta_\delta^{\beta=0}(\varphi_1, \varphi_2) + \\ &\frac{T_k}{2} \tilde{\partial}_t \int_{\tilde{q}} \left\{ \partial_{\varphi_1} \left[\frac{\delta_k^{(10)}(\varphi_1, \varphi_2)}{z_k(\varphi_1)\tilde{q}^2 + s(\tilde{q}^2) + u_k''(\varphi_1)} \right] + \text{perm}(12) \right\}, \end{aligned} \quad (55)$$

where $\beta_\delta^{\beta=0}(\varphi_1, \varphi_2)$ is the zero (auxiliary) temperature beta functional obtained by taking derivatives with respect to φ_1 and φ_2 of the right-hand side of Eq. (36) and we have omitted subdominant terms coming from the difference between \hat{Q}_k and \hat{P}_k (see above).

We can now follow the derivation done for the effect of the (bath) temperature in the formalism without superfields (see paper II²). After changing variables from φ_1, φ_2 to $x = (\varphi_1 + \varphi_2)/2$ and $y = (\varphi_1 - \varphi_2)/2$, one obtains the solution for $\delta_k(x, y)$ in the close vicinity of the fixed point, when both T_k and y go to zero, in the form of a boundary layer

$$\begin{aligned} \delta_k(x, y) &= \\ \delta_{*,0}(x) + T_k &\left[A(x) - B(x) \sqrt{1 + C(x)^2 \left(\frac{y}{T_k} \right)^2} \right] + O(T_k^2), \end{aligned} \quad (56)$$

where $A(x)$, $B(x)$, $C(x)$ are well-behaved functions of x and $A(x)|C(x)|$ is equal to the absolute value of the coefficient of the cusp in the zero-temperature fixed-point function $\delta_*(x, y)$.

The connection between this boundary-layer phenomenon and the presence of rare low-energy excitations above the ground-state known as “droplets”^{69,70} is the same as that already discussed in our previous work² (see also Refs. [4,5]) and the discussion is not repeated here.

VI. CONCLUSION

In this paper and the preceding one¹⁹ we have extended our NP-FRG approach of disordered systems, which was initiated in the previous articles of this series.^{1,2} The objective was to discuss the property of dimensional reduction and its breakdown in the RFIM from the Parisi-Sourlas¹⁵ perspective of an underlying supersymmetry and its breaking. We have reformulated the NP-FRG

in a superfield formalism and, to do so, we have proposed a solution for properly selecting the ground state among the many metastable states that are present at zero temperature. Through the introduction of an appropriate regulator and a supersymmetry-compatible nonperturbative approximation, we have been able to follow the supersymmetry (more precisely, the superrotational invariance) and its spontaneous breaking along the RG flow.

Despite the fact that the effective hamiltonian (or microscopic action) in the presence of a random field has numerous minima in the region of interest near the critical point, dimensional reduction need not be systematically broken. By implementing a NP-FRG flow for the cumulants of the renormalized disorder, our work shows that there is a finite range of dimension below the upper critical dimension, $d_{DR} \leq d \leq d_{uc} = 6$, for which this multiplicity of minima has no effect on the long-distance properties of the model. More precisely, the scaling behavior around the critical point conforms to the dimensional-reduction predictions. The associated fixed point is characterized by a nonanalytic dependence of the effective action in the dimensionless fields, at odds with a naive description based on perturbation theory, but the nonanalyticity is too weak to alter the spectrum of critical exponents. It is only by going below a critical dimension d_{DR} , which we numerically find around 5, that spontaneous breaking of the supersymmetry (superrotational invariance) takes place. The robustness of our theoretical description, which explains the pending puzzles concerning the critical behavior of the RFIM, is supported by the good agreement obtained between the NP-FRG predictions for the critical exponents and the available values from computer studies.

Finally, one may wonder whether the formalism developed here can be useful in a different context. An obvious extension is a study of the long-distance physics of self-avoiding branched polymers and the associated property of reduction to the Yang-Lee edge singularity problem in two fewer dimensions.⁷¹ Dimensional reduction is known to hold in this case,⁷² and this provides a benchmark model to test the ability of our nonperturbative approximation scheme to reproduce this property. Another extension concerns the hysteresis behavior and out-of-equilibrium phase transitions of the RFIM at zero temperature when driven by a slow change of the external magnetic field.^{73–75} Work in those directions is under way.

More challenging, and far more speculative too, is the question known as the “Gribov ambiguity” in the nonperturbative quantization of nonabelian gauge field theories.^{26,77,78} The standard Faddeev-Popov gauge-fixing procedure aims at restricting the functional integral over the gauge field to nonequivalent gauge configurations (*i.e.*, configurations that are not obtainable one from another by a gauge transformation). Unfortunately, the gauge-fixing conditions that respect Lorentz invariance and internal (color) symmetry, such as the Landau

gauge condition, are inconsistent because they admit solutions that are equivalent up to a gauge transformation (“Gribov copies”). There has been a strong theoretical effort to overcome this problem. This “Gribov ambiguity” does not affect the perturbative regime, so that calculations at high energy can be performed in the standard Faddeev-Popov approach. However, very little is known on its influence in the nonperturbative regime. The Gribov-Zwanziger model^{78,79} enables one to reduce the number of copies taken into account in the functional, but not to a single one, and, up to now, there is no unambiguous gauge-fixing procedure. The strong connections between the Faddeev-Popov and the Parisi-Sourlas formalisms, and their common failure when the relevant field equation has a multiplicity of solutions, have been underlined.²⁶ The relevance of the tools and concepts developed in the present and the preceding papers to the problem of the Gribov ambiguity has yet to be investigated.

Appendix A: Stability of the nonanalytic behavior along the RG flow

To illustrate that a linear cusp in $\Gamma_{k2;x_1,x_2}^{(11)}(\phi_1, \phi_2)$ does not generate a supercusp and may break dimensional reduction, we consider the ERGE for the Fourier transform of the 2-point proper vertex $\Gamma_{k2}^{(11)}(q^2; \phi_1, \phi_2)$ (for uniform fields) in the limit $\phi_2 \rightarrow \phi_1$. We switch to the variables $\phi = (\phi_1 + \phi_2)/2$ and $Y = (\phi_1 - \phi_2)/2$ and we study a situation in which a cusp is present:

$$\Gamma_{k2}^{(11)}(q^2; \phi, Y) = \Gamma_{k2;0}^{(11)}(q^2; \phi) + \Gamma_{k2;\alpha}^{(11)}(q^2; \phi) |Y|^\alpha + \dots, \quad (\text{A1})$$

where $Y \rightarrow 0$ and $\alpha < 2$ (recall that $\Gamma_{k2}^{(11)}$ is even in ϕ and Y separately). With the assumption concerning the behavior of the proper vertices detailed in Sec. III-B, we find that the flow of $\Gamma_{k2}^{(11)}$, where for simplicity we take $q^2 = 0$, can be written in the $Y \rightarrow 0$ limit:

$$\begin{aligned} \partial_t \Gamma_{k2}^{(11)}(q^2 = 0; \phi, Y) &\simeq \text{regular} - \frac{\alpha^2}{4} |Y|^{2(\alpha-1)} \times \\ &\tilde{\partial}_t \int_{q'} \hat{P}_k(q'^2; \phi)^2 \Gamma_{k2;\alpha}^{(11)}(q'^2; \phi)^2 - \tilde{\partial}_t \int_{q'} \hat{P}_k(q'^2; \phi) \\ &\times [\partial_Y \Gamma_{k3;-q'q'0}^{(111)}(\phi + Y, \phi + Y, \phi_3)|_{\phi_3=\phi-Y}) \\ &- \partial_Y \Gamma_{k3;-q'q'0}^{(111)}(\phi - Y, \phi - Y, \phi_3)|_{\phi_3=\phi+Y})], \end{aligned} \quad (\text{A2})$$

where *regular* denotes the terms that could be obtained by directly considering the flow of $\Gamma_{k2}^{(11)}(q^2 = 0; \phi, Y = 0)$ and by assuming a regular behavior of its beta-functional when $Y \rightarrow 0$. The contribution due to the derivatives of the third cumulant $\Gamma_{k3}^{(111)}$ is an even function of Y ; the $Y = 0$ term contributes to the regular component of the beta-functional and the next term is a $O(|Y|^{\alpha'})$ with $\alpha' > 2$.

The above equation indicates that a “supercusp” with

$\alpha < 1$ leads to an ill-defined flow for $\Gamma_{k2}^{(11)}(q^2 = 0; \phi, Y = 0)$. For $\alpha \geq 1$ (cusp) one should also study the ERGE for the third cumulant to determine the leading nonanalyticity (*i.e.*, α') and therefore study the stability of the cusp under further RG flow. In any case, Eq. (A2) shows that a cusp weaker than linear, with $1 < \alpha < 2$, has no effect on the flow of $\Gamma_{k2}^{(11)}(q^2 = 0; \phi, Y = 0)$ and, since $2\eta_k - \bar{\eta}_k$ is obtained from the latter, is not expected to modify dimensional reduction. A linear cusp ($\alpha = 1$) on the other hand provides a possible mechanism for the failure of dimensional reduction while not *a priori* generating stronger nonanalyticities. In order to actually lead to breakdown of dimensional reduction, the linear cusp must of course remain in the renormalized second cumulant up to the appropriate fixed point.

Appendix B: Dimensionless threshold functions

The various dimensionless threshold functions correspond to the various 1-loop integrals involving the infrared cutoff functions and the propagators (after account of the scaling dimensions). More specifically, they are defined as follows:^{28,39,40}

$$l_{q_1, q_2}^{(d)}(w_1, w_2; z_1, z_2) = -\frac{1}{2} \int_0^\infty dy y^{d/2-1} \tilde{\partial}_t \left\{ \frac{1}{(p_1(y) + w_1)^{q_1} (p_2(y) + w_2)^{q_2}} \right\}, \quad (\text{B1})$$

$$m_{q_1, q_2}^{(d)}(w_1, w_2; z_1, z_2) = -\frac{1}{2} \int_0^\infty dy y^{d/2-1} \tilde{\partial}_t \left\{ \frac{y (\partial_y p_1(y))^2}{(p_1(y) + w_1)^{q_1} (p_2(y) + w_2)^{q_2}} \right\}, \quad (\text{B2})$$

$$n_{q_1, q_2}^{(d)}(w_1, w_2; z_1, z_2) = -\frac{1}{2} \int_0^\infty dy y^{d/2-1} \tilde{\partial}_t \left\{ \frac{y \partial_y p_1(y)}{(p_1(y) + w_1)^{q_1} (p_2(y) + w_2)^{q_2}} \right\}, \quad (\text{B3})$$

with $p_a(y) = y(z_a + r(y))$, $a = 1, 2$, $y = q^2/k^2$, and $\tilde{\partial}_t$ acts only on the dimensionless function $r(y) = s(y)/y = \hat{R}_k(yk^2)/Z_k yk^2$ that is contained in the $p_a(y)$'s^{27,28,40} (by definition, $\partial_t r(y) = -[\eta r(y) + 2yr'(y)]$). Note that, in addition to the dependence on $w \equiv u_k''(\phi)$ and $z \equiv z_k(\phi)$, the threshold functions explicitly depend on the scale k via the running anomalous dimension η_k . (Note also that the functions $m^{(d)}$ and $n^{(d)}$ are not symmetric in the exchange of the indices 1 and 2.) The generic properties of these dimensionless threshold functions are discussed in detail in Ref. [28,39,40].

In the present study, it is also necessary to introduce additional threshold functions, which accounts for the fact that the cutoff function \tilde{R} has an anomalous scaling

compared to \hat{R} when dimensional reduction is broken. These threshold functions then always appear multiplied by $(\eta - \bar{\eta})$ in the flow equations. We define:

$$j_{q_1, q_2}^{(a, d)}(w_1, w_2; z_1, z_2) = -\int_0^\infty dy y^{d/2-1} \frac{(r(y) + yr'(y))^a}{(p_1(y) + w_1)^{q_1} (p_2(y) + w_2)^{q_2}} \quad (\text{B4})$$

and

$$h_q^{(d)}(w; z) = -\int_0^\infty dy y^{d/2} \frac{r(y) + yr'(y)}{(p(y) + w)^q} \left\{ \frac{2z + r(y) + yr'(y)}{2} - \frac{qzy}{p(y) + w} (z + r(y) + yr'(y)) \right\}. \quad (\text{B5})$$

We summarize here a number of relations that are useful in the developments of Sec. IV. One has $l_q^{(d)}(w; z) = l_{q_1, q_2}^{(d)}(w, w; z, z)$, $m_q^{(d)}(w; z) = m_{q_1, q_2}^{(d)}(w, w; z, z)$, $n_q^{(d)}(w; z) = n_{q_1, q_2}^{(d)}(w, w; z, z)$, $j_q^{(a, d)}(w; z) = j_{q_1, q_2}^{(a, d)}(w, w; z, z)$, with $q_1 + q_2 = q$. In addition, the functions $n_q^{(d)}$'s and $l_q^{(d)}$'s are related by

$$n_q^{(d)}(w; z) = \frac{d}{2(q-1)} l_{q-1}^{(d)}(w; z). \quad (\text{B6})$$

From the above definitions, one also straightforwardly finds that

$$\partial_{w_1} l_{q_1, q_2}^{(d)}(1, 2) = -q_1 l_{q_1+1, q_2}^{(d)}(1, 2), \quad (\text{B7a})$$

$$\partial_{z_1} l_{q_1, q_2}^{(d)}(1, 2) = -q_1 l_{q_1+1, q_2}^{(d+2)}(1, 2), \quad (\text{B7b})$$

and

$$\partial_{w_1} m_{q_1, q_2}^{(d)}(1, 2) = -q_1 m_{q_1+1, q_2}^{(d)}(1, 2), \quad (\text{B8a})$$

$$\partial_{z_1} m_{q_1, q_2}^{(d)}(1, 2) = -q_1 m_{q_1+1, q_2}^{(d+2)}(1, 2) + \frac{d}{q_1-1} l_{q_1-1, q_2}^{(d+2)}(1, 2), \quad (\text{B8b})$$

$$\partial_{z_2} m_{q_1, q_2}^{(d)}(1, 2) = -q_2 m_{q_1, q_2+1}^{(d+2)}(1, 2). \quad (\text{B8c})$$

Similar relations hold for the functions $j_{q_1, q_2}^{(a, d)}$ and $h_q^{(d)}$. From these expressions one can obtain the derivatives with respect to the field arguments φ_a by using that $w \equiv u_k''(\phi)$ and $z \equiv z_k(\phi)$.

Appendix C: Some technical aspects of the numerical resolution

1. Optimization of the IR cutoff function

In the exact formulation of the NP-FRG, the results of the flow equations are independent of the IR regulator, which is only an intermediate means that does not

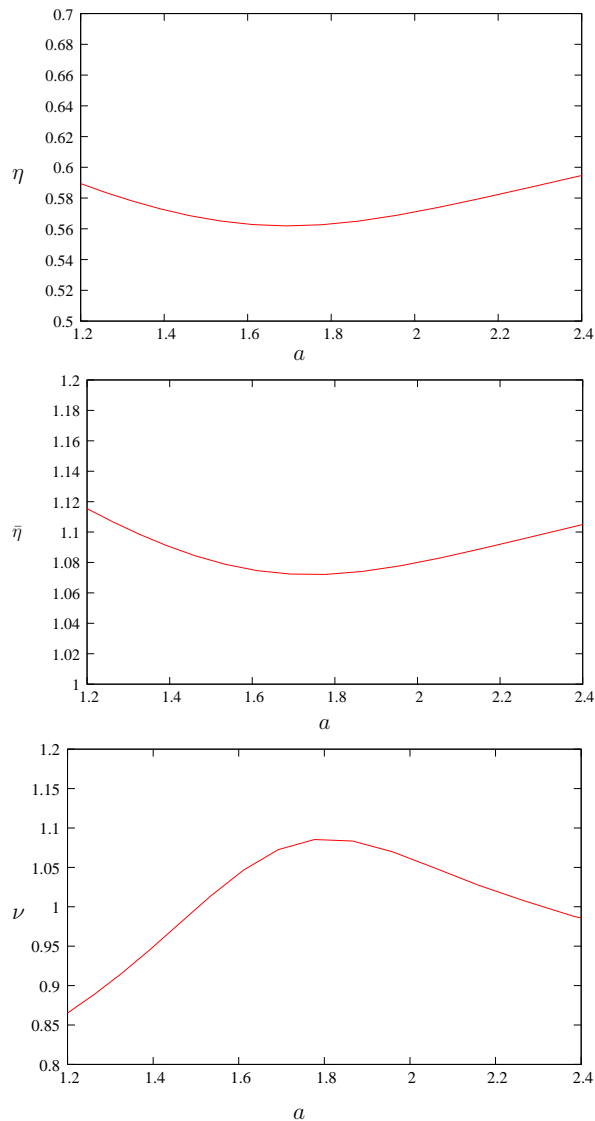


FIG. 11: Dependence of the anomalous dimensions η (upper panel), $\bar{\eta}$ (middle panel) and of the critical exponent ν (lower panel) on the parameter a of the IR cutoff function in Eq. (51). The “principle of minimal sensitivity” leads to a determination of the anomalous exponents $\eta = 0.565$, $\bar{\eta} = 1.075$, $\nu = 1.08$ (at the minima). The variations of the anomalous dimensions are very small over a wide range of a and enables us to evaluate the precision to $\Delta\eta = 0.03$ and $\Delta\bar{\eta} = 0.04$. ν is more sensitive to the regulator parameters and we estimate $\Delta\nu = 0.2$

affect the long-distance physics. Once approximations

are introduced, a residual dependence on the choice of regulator however remains. (A similar situation occurs in perturbation theory where a residual dependence on the parameters of the Borel resummation procedure is observed.) One can in some sense optimize the choice of IR cutoff function by demanding that the output, say the critical exponents, satisfies a property of “minimal sensitivity” such that by varying the characteristics of the cutoff function around the optimum, a minimal variation of the exponents results. This guarantees the stability of the results. We apply this procedure to the type of function given in Eq. (51) where the parameters a , b , and c can be varied.

We illustrate in Fig. 11 the dependence of the anomalous dimensions in $d = 3$ on the parameter a when b and c are kept fixed at the values 0.81 and 0.14 respectively (these latter values were determined by preliminary variational studies to find minimal sensitivity). One can see that there is a large domain of values in which η and $\bar{\eta}$ vary little (say by less than 5%). We use the “principle of minimal sensitivity” to determine the best estimate of the critical exponents [note that the minima of $\eta(a)$ and $\bar{\eta}(a)$ happen at very close values of a]. We also estimate the error bars from the observation that when a varies in the range $[1.5, 3]$, the maximum variation of η is of 0.03 and that of $\bar{\eta}$ is of 0.04. These are the values that we have reported in Sec. V.

2. Regularization of the numerical instabilities

During the numerical integration of the flow, we encountered numerical instabilities in physically unimportant regions (typically, at large fields). The origin of these instabilities can be understood as follows. In the region of large fields, the threshold functions rapidly decrease to zero. In this limit, the flow of a function is given by the dimensional part, *i.e.* the flow obtained by setting $v_d \equiv 0$ [*e.g.* $\partial_t u'_k(\varphi) \sim -\frac{1}{2}(d - 2\eta_k + \bar{\eta}_k)u'_k(\varphi) + \frac{1}{2}(d - 4 + \bar{\eta}_k)\varphi u''_k(\varphi)$]. This flow no longer depends on the second derivative of the associated function with respect to the field. However, this diffusive-like term [$u'''_k(\varphi)$ in our previous example] is very important to stabilize the numerical integration. In practice, when necessary, we have therefore added by hand a small diffusive contribution to the flow equations and checked that the numerical results are stable under varying the strength of these extra terms.

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unlikely to be captured by such an approach. In this case the rare regions that are involved are exponentially suppressed as their size increases and produce very weak (essential) singularities in the thermodynamics. Such a behavior does not show up easily once the average over disorder

has been performed, as done in the cumulants. Droplets on the other hand are “power-law rare” regions in their size and have a clear signature in the functional dependence of the cumulants of the renormalized disorder (see *e.g.* the detailed work by Balents and Ledoussal on the random manifold model.^{4,5}) The same is true for “avalanches”, which are collective discontinuous events occurring in *typical* samples at zero temperature for exceptional values of the applied source.

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